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Optimal allocation mechanisms with type-dependent negative externalities

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Abstract I analyze optimal auction design in the presence of linear type-dependent negative externalities. I characterize the properties of the optimal mechanism when externalities are "strongly decreasing" and "increasing" in the agent's valuation and I discuss its implementation with sealed-bid auctions. Interestingly, bidding strategies are not necessarily increasing in valuations, and the optimal mechanism can be implemented by setting a price ceiling instead of a reserve price.

Keywords Auctions · Type-dependent externalities · Mechanism design

JEL Classification D44 · D62

1 Introduction

Consider the following examples: two pharmaceutical laboratories competing for the acquisition of a license, two firms competing for the delegation of a procurement contract, and two employees competing for a promotion. In all these cases, the ex-post utility of the "winner" is higher and that of the "loser" is lower than in the status quo scenario. The laboratory without the license will face a tougher competition on the product market. The firm without the contract will suffer a reputation loss

This paper builds on two cases analyzed in Brocas (2001–2009).

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relative to its competitor. The employee without the promotion will have to wait for new opportunities which may not arise in the short term. In all these examples, the winner induces a *negative externality* on the loser. This has two implications. First, each agent is willing to incur a payment not only to obtain the good but also to prevent the owner from selling it to the rival. Second, in order to be optimal, the pricing and allocation mechanism must be modified accordingly.

Allocation mechanisms in the presence of negative externalities have been studied in many articles. In a licensing context, Katz and Shapiro (1986) argue that the optimal price is affected by the ex-post asymmetric competition in the product market between the firms that obtain a patent and those that do not. As Kamien et al. (1992) show, the seller can extract some payments even from the firms that do not obtain the license. Jehiel et al. (1996, 1999) analyze the optimal allocation mechanism when agents have private information about their valuation for the good. In the presence of negative externalities, if the owner decides to sell the item to one agent, his competitors are ready to pay an amount equal to the imposed externalities in order to prevent that sale. Therefore, in equilibrium, the seller may prefer to collect those payments and keep the item. As the authors show, the mechanism that maximizes the revenue of the seller is an auction with the same qualitative properties as the standard auction without externalities.[1] First, it can be implemented with a first or second price sealed-bid auction with a reserve price in which bids are increasing in valuations. And second, the mechanism is ex-post inefficient, in the sense that some profitable trades do not occur. The mechanism exhibits only a few differences compared to an auction without externalities. In particular, the seller needs to resort to an entry fee and, as the size of the externality increases, both the entry fee and the reserve price increase. At equilibrium, the seller extracts more revenue from the auction and the good is sold with a smaller probability.

In these papers, the externality is independent from the agent's valuation.[2] Yet, in many situations, valuations, and externalities are highly correlated. Consider the licensing example. A firm with a large market may also have a greater capacity to exploit the innovation than its smaller counterparts. Therefore the large firm has a relatively higher valuation for the license, and the higher the market share, the higher this valuation. At the same time, the large firm is more likely to drive other firms out of the market if it obtains the license, and induces negative externalities on its competitors. It is even plausible that the larger the initial market share is, the higher those externalities. However, if a competitor ever gets the license, the large firm suffers externalities too. If the innovation is minor, the large firm will still remain in business, and the higher the initial market share, the lower the externality suffered. By contrast, if the innovation is drastic, the large firm will be driven out of the market. In that case, the higher the initial market share, the higher the loss, and therefore the higher the externality. This

[1] See Myerson Myerson (1981) for the seminal paper on optimal auctions and Engelbrecht-Wiggans (1980), McAfee and McMillan (1987) and Klemperer (1999) for surveys.

[2] In Jehiel et al. (1996), the size of the externality suffered by an agent depends only on the identity of the winner and not on his valuation. In Jehiel et al. (1999), the externality suffered by an agent is depended on the identity of the winner, but it is not related to his valuation. Other mechanism design problems consider identity-dependent externalities (e.g., Aseff and Chade 2008 for the case of multi-unit auctions).

suggests that valuation for the good, suffered externality and imposed externality are linked through underlying variables: the specific structure of the industry determines the sign and amount of the correlation.

The purpose of this paper is to show that the details of these correlations matter to design allocation mechanisms. We are interested in characterizing the optimal allocation mechanism when externalities are type-dependent and showing that different relationships between valuations and externalities result in different qualitative properties. Furthermore, the procedures (e.g., sealed bid auctions) that implement the optimal outcomes may also vary, and be quite different from what has been already suggested in the literature.[3]

From a general viewpoint, incorporating type-dependent negative externalities modifies the optimal design of the auction in several ways. First, the reservation utility of each agent depends on the size of the externality he suffers if a rival obtains the good, and therefore on his valuation.[4] Second, each agent must be induced to disclose his willingness to pay to obtain the good (as in the standard theory) and his willingness to pay to prevent a rival from getting it. Since the incentives in terms of informational rents needed to fulfill these two goals are sometimes in conflict, the incentive-compatibility constraint will not necessarily specify an equilibrium utility monotonic in the agent's valuation, which departs from standard problems.[5] Last, the seller's objective is to extract the valuation for the object but also to price the externality. This impacts the final allocation as well as the final prices.

The paper shows that the optimal mechanism will depend on whether the externality suffered by an agent is "increasing" (case 1), or "strongly decreasing" (case 2) in his valuation. As reviewed in the next paragraphs, each case exhibits a different departure and some novel properties relative to the standard auction problem.

When the externality suffered by the agent increases with his valuation (case 1), the equilibrium utility of the agent in the optimal auction is not necessarily monotonic in his type. Given participation, medium types obtain lower utility compared to small or high types. Suppose that agent 1 has the lowest possible valuation, in which case he also suffers a small externality. It is optimal for the seller to either keep the good or allocate it to agent 2. If the valuation of agent 1 increases marginally, both his willingness to pay to obtain the good and his willingness to pay to prevent a sale to his rival increase. The seller is now relatively more willing to keep the good in exchange of a payment from agent 1. If this happens, agent 1 still avoids the sale but ends up with a lower utility. Naturally, if agent 1's valuation increases sufficiently, then his chances of obtaining the good himself start growing and so does his overall utility.

[3] Note that similar situations have been investigated by Jehiel and Moldovanu (2000) when the item is sold via particular procedures. However, the main focus of the present paper is to characterize optimal procedures. Also, Carrillo (1998), and Figueroa and Skreta (2009) studied optimal allocations with type-dependent externalities in related settings. Those analyses will be reviewed below.

[4] Optimal contracting under type-dependent reservation utilities has been analyzed in Lewis and Sappington (1989), Maggi and Rodriguez (1995), Jullien (2000) for the one-agent case. The multi-agent case has been investigated in Carrillo Carrillo (1998), Brocas (2009, 2011), and Figueroa and Skreta (2009).

[5] Our claim here is *not* that informational rents may be non-monotonic (a feature standard in the presence of type-dependent reservation utilities and countervailing incentives). Instead, we argue that the total equilibrium utility may be non-monotonic. This also occurs in Parlane (2001) but for a different reason.

Moreover, at equilibrium, the good is allocated to the agent with the highest valuation provided it exceeds a reserve price. The reserve price faced by each agent is increasing in the valuation of the rival: as the valuation of the rival increases, the externality the rival may suffer also increases, and it becomes relatively more profitable to extract a payment from the rival in exchange of keeping the good (Proposition 1).

When the externality suffered by the agent strongly decreases with his valuation (case 2), it is never optimal to allocate the good to the agent with the highest valuation. This is the case because the lowest valuation agent is willing to pay more to prevent the sale to his rival compared to what the rival is willing to pay to acquire the item. This means that the optimal strategy for the auctioneer is either to sell the good to the agent with lowest valuation or to extract rents from the agents for not selling the good at all. Furthermore, the reserve price faced by each agent is now decreasing in the valuation of the rival: as the valuation of the rival increases, the externality the rival may suffer decreases, and it becomes relatively more profitable to sell the good to the agent rather than to extract a payment from the rival in exchange of keeping the good. Last, compared to the full information case, it is optimal to increase the overall likelihood of selling the good to decrease informational rents, which also departs from standard results in the auction literature (Proposition 2).

We show that these mechanisms can be implemented with a second price sealed bid auction that specifies for each agent an entry fee and a reserve price contingent on the bid of the rival (Proposition 4). One interesting feature is that, when externalities are strongly decreasing in the valuation of the agent (case 2), then the bid of an agent is decreasing in his valuation. The idea is simply that an agent with a low valuation has a much higher willingness to pay to prevent a sale to the rival than an agent with a high valuation. As a result, he is willing to submit a higher bid because obtaining the good is an insurance against suffering the externality. More generally, in second price sealed bid auctions, the bidding strategy of an agent is decreasing in his own valuation when externalities are strongly decreasing (Proposition 3). It also implies that in the optimal mechanism, the reserve price is a price ceiling in that case. The seller commits herself not to accept high bids (submitted by low valuation agents) in order to collect large payments made as an insurance to prevent competitors from getting the good.

Carrillo (1998) is the first paper to study the optimal contract with multiple agents and type-dependent externalities.[6] The article shows that the good will be allocated more (resp., less) often than under full information if the reservation utility strongly (resp., weakly) increases with the agent's valuation. The model looks at two situations: one similar to our case 2 (with decreasing informational rents) and another one similar to the fixed externality case (with increasing informational rents, treated as a special case hereafter). However, contrary to our case 2 and due to the restrictions imposed on the externality, in Carrillo (1998) the good is never allocated to the agent with lowest valuation. More recently, Figueroa and Skreta (2009) have also studied optimal auctions with type-dependent externalities. However, as Carrillo (1998), the authors restrict the attention to functional forms delivering an increasing equilibrium

[6] Jehiel and Moldovanu (2000) also consider multiple agents and type-dependent externalities. However, they restrict their attention to a specific externality and do not analyze the optimal mechanism.

utility (which rules out our case 1), and look at three situations: two similar to Carrillo Carrillo (1998) and one resulting from the coexistence of two possible outside options, an issue we do not address in this paper.[7] Overall, none of these articles study the allocation when the externality is increasing in the valuation (case 1) and they analyze a different version of the case where the externality is decreasing in the valuation (case 2) delivering different properties. Last, none of these studies address implementation.

The paper is organized as follows. Section 2 presents the general model and determines the constraints that the optimal mechanism must satisfy. Sections 3 and 4 characterize the properties of the optimal mechanism when externalities are increasing and strongly decreasing in the valuation, respectively. Section 5 discusses second price sealed bid auctions. Finally, Sect. 6 concludes. Proofs are relegated to the appendix.

2 A simple allocation problem

2.1 The model

A seller offers one indivisible good to two risk-neutral potential buyers 1 and 2, indexed by i and j. Buyer i (he) derives utility v_i when he gets the good (or wins). We will call v_i, his "willingness to pay", "type" or "valuation" and $v = (v_i, v_j)$ the vector of valuations of both agents. Each valuation is drawn independently from a known distribution with c.d.f. $F(v_i)$ and density $f(v_i)$. $F(\cdot)$ is strictly increasing and continuously differentiable on the interval $[\underline{v}, \overline{v}]$, with $0 \leq \underline{v} < \overline{v}$. Also, $F(\underline{v}) = 0$ and $F(\overline{v}) = 1$. The valuation for the good of the seller (she) is zero.

To address the economic problems highlighted in the introduction, we assume that buyer i suffers a negative externality if his rival obtains the good. Formally, we assume buyer i suffers an externality $-\alpha_i(v)$ when buyer $j \neq i$ obtains the good. This externality is negative and depends on both valuations. In order to keep the analysis as tractable as possible, we restrict to linear externalities:

Assumption 1 $\alpha_i(v) = \alpha_a v_i + \alpha_b v_j + \gamma > 0 \quad \forall\, v_i, v_j$ and $(\alpha_a, \alpha_b, \gamma) \in \mathbb{R}^3$.

In this game, each agent faces three possible outcomes: (i) he obtains the good, (ii) nobody obtains the good, and (iii) the rival obtains the good. In the absence of externalities, (ii) and (iii) are identical from each agent's viewpoint. This implies that the eventual allocation is irrelevant for an agent who decides not to participate in the auction. In the presence of negative externalities however, (ii) and (iii) are different and it becomes important to specify a rule in case one agent decides not to participate. We endow the seller with full commitment power, and in particular with the ability to commit to any such rule.

[7] The point of the paper is to study the role of optimal threats and show that when several outside option coexist, the seller must randomize between them.

2.2 Examples

Let us consider a typical technology allocation problem. A new technology is available for sale and two firms A and B compete to adopt it. We consider three distinct market situations.[8]

In the first situation, firms A and B compete with the same efficiency parameter e. Profits are $\phi(e,e)$. If A adopts the new technology, its efficiency parameter becomes $\theta_A\,(>e)$. Given B's efficiency parameter is still e, A's profit is now $\phi(\theta_A,e)>\phi(e,e)$. Similarly, if B adopts the new technology, its efficiency becomes $\theta_B(>e)$ and its profit is $\phi(e,\theta_B)$. In that setting A's valuation is simply $\phi(\theta_A,e)-\phi(e,e)\equiv v_A$ and A's externality is $\phi(e,\theta_B)-\phi(e,e)\equiv\alpha_A(v_B)$. The externality suffered by firm A is a function of the valuation of firm B only. This property is captured qualitatively by our linear setting when $\alpha_a=0$ and $\alpha_b\neq 0$.[9]

In the second situation, firms A and B are ex ante monopolists on markets A and B, respectively. Their respective efficiency parameters are θ_A and θ_B and their respective profits are $\phi_A(\theta_A)$ and $\phi_B(\theta_B)$. If A gets the license, it drives B out of market B. If B gets the license, it drives A out of market A. In that case, A's valuation is $\phi_B(\theta_A)\equiv v_A$ and A's externality is $\phi_A(\theta_A)\equiv\alpha_A(v_A)$. The externality suffered by firm A is a function of the valuation of firm A only. We capture this property when $\alpha_a\neq 0$ and $\alpha_b=0$. Furthermore, assuming that profit functions are increasing in efficiency parameters, we have $\theta_A=\phi_B^{-1}(v_A)$ and $\alpha_A(v_A)=\phi_A(\phi_B^{-1}(v_A))$ which corresponds to the linear case $\alpha_a\geqslant 0$.[10]

In the third situation, firms A and B are again ex ante monopolists respectively on markets A and B, with efficiency parameters θ_A and θ_B and profits $\phi_A(\theta_A)$ and $\phi_B(\theta_B)$. Now, if A gets the license, market A expands while market B shrinks. The new profits are $\overline{\phi}_A(\theta_A)\geqslant\phi_A(\theta_A)$ and $\underline{\phi}_B(\theta_B)\leq\phi_B(\theta_B)$. Suppose that $\overline{\phi}_i'\geqslant\phi_i'$ and $\underline{\phi}_i'\leq\phi_i'$ and let $\Phi_i=\overline{\phi}_i-\phi_i$. Then, A's valuation is $v_A=\Phi_A^{-1}(\theta_A)$ and A's externality is $\alpha_A(v_A)=\phi_A(\Phi_A^{-1}(v_A))-\underline{\phi}_A(\Phi_A^{-1}(v_A))$. This example corresponds to the linear approximation $\alpha_a<0$.[11]

Note that the case $\alpha_a\geqslant 0$ corresponds to a drastic innovation. The firm that has more interest in obtaining the license is also the one that has more to lose if the license is allocated to the rival. By contrast, in the case $\alpha_a<0$, the innovation is minor and the firm that has more interest in obtaining the license has less to lose if the license is

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8 Even though we decided to restrict the attention to negative externalities, the analysis could be extended to positive type-dependent externalities. For instance, the private value case of auctions with cross-shareholding (see Dasgupta and Tsui 2004) could be captured by externalities of the form $\alpha_i(v)<0$, $\alpha_a=0$ and $\alpha_b>0$. Incentive collusive transfers in bidding rings (see McAfee and McMillan 1992) generate similar externalities.

9 Assuming profits are increasing in efficiency parameters, the example corresponds to $\alpha_b\geqslant 0$.

10 Note that the profits ϕ_A and ϕ_B are defined up to a constant and we can always find a parametrization such that $\alpha_A(v_A)<0$ for all v_A.

11 Given we concentrate on situations in which the externality is negative at each point, profit functions must be such that $\phi_A(\Phi_A^{-1}(\overline{v}))-\underline{\phi}_A(\Phi_A^{-1}(\overline{v}))>0$. This is captured in our linear setting by assuming $\gamma>-\alpha_a\overline{v}$.

allocated to the rival. The lower α_a, the less a firm with high v_i suffers when its rival adopts the new technology.

2.3 Allocation under complete information

Call $\pi_i^F(v)$ agent i's *net surplus* under full information. It represents the difference in the payment that the seller can extract from i in exchange of the good and the payment she can extract from j in exchange of not selling the good to i:

$$\pi_i^F(v) = v_i - \alpha_j(v) \quad \text{for all } i, j = 1, 2 \text{ and } i \neq j$$

where, by symmetry, $\pi_j^F(v_i, v_j) \equiv \pi_i^F(v_j, v_i)$. Call also $X_i^F(v)$ the probability of selling the good to agent i under full information. The optimal allocation rule is:

$$X_i^F(v) = 1 \text{ if and only if } \pi_i^F(v) > \max\{0, \pi_j^F(v)\} \quad \text{for all } i, j = 1, 2 \text{ and } i \neq j$$

This rule constitutes a benchmark for comparison with the asymmetric information case. Note that the good is not always allocated under complete information ($X_1^F(v) = X_2^F(v) = 0$ when $\pi_1^F(v) < 0$ and $\pi_2^F(v) < 0$). However, the seller still collects payments: agent 1 pays $\alpha_1(v)$ in exchange of not selling the good to 2 and agent 2 pays $\alpha_2(v)$ in exchange of not selling the good to 1. The seller does not sell the good in order to collect those payments when they exceed the payments she can obtain from selling the good (namely $v_i + \alpha_i(v)$ from agent i if she allocates the good to i).[12] Last, under complete information it is optimal to allocate the good to the agent with the highest valuation when $1 + \alpha_a > \alpha_b$, and to the agent with the lowest valuation when $1 + \alpha_a < \alpha_b$, provided the surplus derived from these sales are positive.

2.4 Feasible mechanisms under asymmetric information

The reservation utility of each agent is given by the outcome of the auction if he does not show up and it is mechanism dependent. Note that an agent wants not only to acquire the good, but also to avoid the externality that results when the rival gets it. Then, he is prone to pay and enter the auction if participating buys him a chance to prevent the other agent from acquiring the good. This generates rents that can be captured by the seller. They are maximized when every agent enters which is guaranteed if the seller can commit to give the good for free to one agent if the other does not participate. This is well-known and follows directly from the facts that the seller has full commitment power and externalities are negative.[13] Overall, the *reservation utility* of bidder i is

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[12] Said differently, the seller allocates the good to agent i rather than j if his net surplus $v_i + \alpha_i(v)$ is the highest, provided the net surplus exceeds the externality payments that can be collected by not selling at all $\alpha_i(v) + \alpha_j(v)$.

[13] Although standard in the literature on auctions with externalities (see e.g., Carrillo 1998, Jehiel et al. 1996, 1999 etc.), this assumption is strong. If an agent does not show up, the seller will have ex-post incentives to conduct the auction with only one bidder rather than give him the good for free. In Brocas (2003),

$$w_i(v_i) = -\alpha_a v_i - \alpha_b \int_{\underline{v}}^{\overline{v}} v_j\, f(v_j)\mathrm{d}v_j - \gamma. \tag{1}$$

The revelation principle applies in our setting and we can restrict the attention to direct mechanisms that induce truth-telling. A direct mechanism is characterized by the interim probability that agent i gets the good, $X_i(v_1, v_2)$ and the associated transfers $t_i(v_1, v_2)$. Let $u_i(v_i, v_i')$ be the *expected utility* of bidder i when he participates in the auction, his valuation is v_i, he announces v_i', and the other bidder discloses his true valuation v_j. We also denote by $u_i(v_i) \equiv u_i(v_i, v_i)$ his expected utility under truthful revelation. We have

$$u_i(v_i, v_i') = E_{v_j}\Big[v_i X_i(v_i', v_j) - \alpha_i(v_i, v_j)X_j(v_i', v_j) - t_i(v_i', v_j)\Big]. \tag{2}$$

The mechanism must satisfy the following three constraints. First, agents must prefer to participate in the auction rather than not (individual-rationality):

$$u_i(v_i) \geqslant w_i(v_i) \quad \forall\, i, v_i.$$

Second, they must be better-off by disclosing their true valuation (incentive-compatibility).

$$u_i(v_i) \geqslant u_i(v_i, v_i') \quad \forall\, i, v_i, v_i'.$$

Third, the selection rule must be feasible:

$$X_1(v) \geqslant 0, \quad X_2(v) \geqslant 0, \quad X_1(v) + X_2(v) \leq 1 \quad \forall\, v.$$

Lemma 1 *In the optimal mechanism, the seller solves the following program $\mathcal{P}$:*

$$\mathcal{P} : \max \int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{\overline{v}} \Big[t_1(v) + t_2(v)\Big] f(v_1) f(v_2)\mathrm{d}v_1\mathrm{d}v_2$$

$$\text{s.t. } u_i(v_i) - u_i(v_i') = \int_{v_i'}^{v_i} E_{v_j}\Big[X_i(s, v_j) - \alpha_a X_j(s, v_j)\Big]\mathrm{d}s \quad \forall\, i,\ v_i' \leq v_i \quad (\mathrm{IC}_1)$$

$$E_{v_j}\Big[X_i(v_i', v_j) - \alpha_a X_j(v_i', v_j)\Big] \leq E_{v_j}\Big[X_i(v_i, v_j) - \alpha_a X_j(v_i, v_j)\Big] \quad \forall\, i,\ v_i' \leq v_i \quad (\mathrm{IC}_2)$$

$$u_i(v_i) \geqslant w_i(v_i) \quad \forall\, i,\ v_i \quad (\mathrm{IR})$$

$$X_i(v_i, v_j) \geqslant 0 \quad \forall\, i,\ v_i,\ v_j \quad (\mathrm{F}_0)$$

$$X_1(v_i, v_j) + X_2(v_i, v_j) \leq 1 \quad \forall\, v_i,\ v_j \quad (\mathrm{F}_1)$$

Footnote 13 continued

we show that when this assumption is relaxed, there is a coordination problem in the behavior of agents that gives rise to multiple equilibria.

Proof See Appendix A-1.

These are the by now standard conditions in mechanism design problems.[14] However, two new features result from the type-dependency of the externality. First, the r.h.s. of inequality (IR) is type-dependent: the reservation utility $w_i(v_i)$ is an (increasing or decreasing) function of the agent's valuation, depending on the slope of the externality. Second, the r.h.s. of equality (IC_1) is not necessarily positive: conditional on accepting to participate in the auction, it is not necessarily the case that agents with higher valuation get a higher equilibrium utility. Contrary to the standard framework, $E_{v_j}[X_i(v_i, v_j) - \alpha_a X_j(v_i, v_j)]$ may or may not be positive. We have

$$\frac{\mathrm{d}}{\mathrm{d}v_i} u_i(v_i) = E_{v_j}[X_i(v)] - \alpha_a E_{v_j}[X_j(v)] \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}v_i} w_i(v_i) = -\alpha_a. \tag{3}$$

We call *informational rents*, $\Phi_i(v_i)$, the difference between the utility of an agent who participates in the auction and his reservation utility. We have

$$\Phi_i(v_i) = u_i(v_i) - w_i(v_i) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}v_i}\Phi_i(v_i) = E_{v_j}[X_i(v)] + \alpha_a(1 - E_{v_j}[X_j(v)]). \tag{4}$$

Given the seller wants to minimize the rent $\Phi_i(v_i)$, there is at least one type $\hat{v}$, which we call a *binding type* for which the (IR) constraint binds: $\Phi_i(\hat{v}) = 0$, that is $u_i(\hat{v}) = w_i(\hat{v})$.[15]

Lemma 2 *In an auction with negative type-dependent externalities the incentive-compatibility and individual-rationality constraints are endogenously linked:*

(i) *When $\alpha_a > 0$, $w_i(v_i)$ is decreasing in v_i, $u_i(v_i)$ may not be monotonic in v_i and $\hat{v} = \underline{v}$.*
(ii) *When $\alpha_a = 0$, $w_i(v_i)$ is constant, $u_i(v_i)$ is increasing in v_i and $\hat{v} = \underline{v}$.*
(iii) *When $\alpha_a \leq -1$, $w_i(v_i)$ is increasing in v_i, $u_i(v_i)$ is increasing in v_i and $\hat{v} = \overline{v}$.*
(iv) *When $\alpha_a \in (-1, 0)$, $w_i(v_i)$ is increasing in v_i, $u_i(v_i)$ is increasing in v_i and $\hat{v} \in [\underline{v}, \overline{v}]$.*

Proof See Appendix A-2.

When the externality increases with the agent's valuation, the utility of the agent if he participates in the auction may increase or decrease with his type. It is always increasing when $\alpha_a = 0$ [case (ii)], and it is not monotonic when $\alpha_a > 0$ [case (i)]. However, even when the utility decreases, the reservation utility decreases faster because the externality is always suffered in case of not participating. As a result, in the absence of adequate incentives, the agent is inclined to *under-state* his type. Therefore,

14 As a reminder, (IC_1) is the (first-order) local optimality condition which ensures that stating the true valuation $v_i' = v_i$ is a local optimum. (IC_2) is the (second-order) monotonicity condition. It ensures the convexity of the equilibrium utility, and therefore that the local optimum is a global maximum.

15 At this stage, we cannot establish whether the binding type is unique or not.

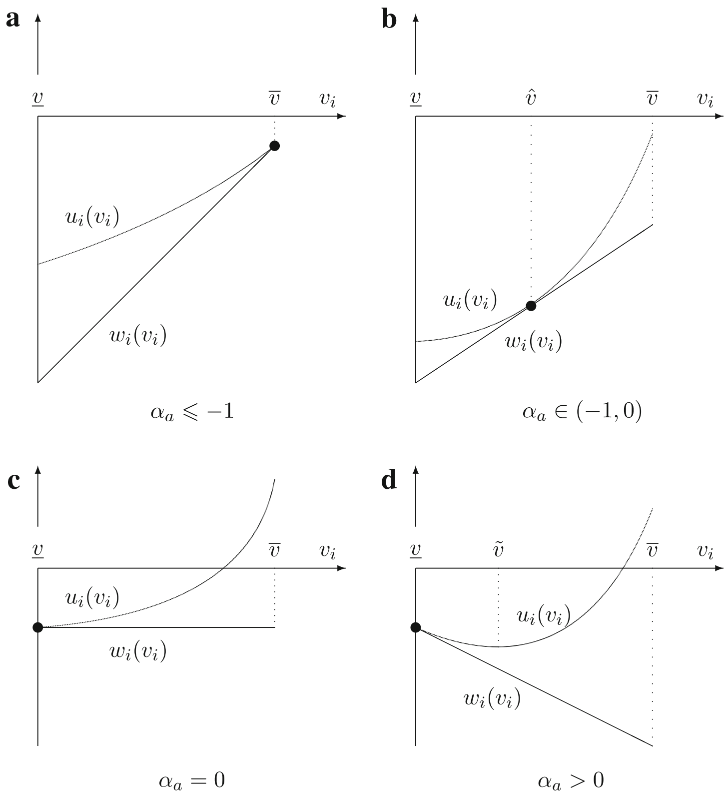

Fig. 1 Equilibrium utility $u_i(v_i)$ and reservation utility $w_i(v_i)$ in the different cases

the informational rents that the seller must leave to the agent to avoid such behavior and induce truth-telling must increase with v_i and the binding type is $\underline{v}$ [cases (i) and (ii)]. When the externality decreases with the agent's valuation, both the utility of participating and the reservation utility increase with the agent's type. Again, since the externality is always suffered under no-participation, the reservation utility increases faster than the utility under participation if the slope is sufficiently steep (formally, if $\alpha_a \leq -1$). In that case and again without the proper incentives, the agent will *overstate* his type, so the informational rents left to induce truth-telling must decrease with v_i [case (iii)]. Last, when the externality is decreasing but small ($\alpha_a \in (-1, 0)$), there exist countervailing incentives to misreport and, as a result, informational rents are not monotonic in the type [case (iv)]. These qualitative properties are illustrated in Fig. 1.

The case $\alpha_a = 0$ [case (ii)] corresponds to the model analyzed by Jehiel et al. (1996) and corresponds also to the first case in Carrillo (1998) and the first case in Figueroa and Skreta (2009), so we will only treat this as a particular case in Remark 1. The problem is more interesting when $\alpha_a \neq 0$, as the reservation utility is type-dependent *and* the equilibrium utility is not necessarily increasing in the agent's type.

Type-dependent reservation utilities have been analyzed in many settings (see e.g., Lewis and Sappington 1989, Maggi and Rodriguez 1995, and Jullien 2000 for the single agent case and Carrillo 1998 for the multi-agent case). This literature has already emphasized that, depending on the relative degree of convexity of the reservation utility and the equilibrium utility, the binding type is at the bottom, at the top or at an interior point. For each of these cases, the informational rents are increasing, decreasing and U-shaped, respectively. Case (iii) falls roughly in the category of the scenarii analyzed in the literature. In particular, it shares common features with the second case analyzed in Carrillo (1998) and the second case analyzed in Figueroa and Skreta (2009). However, some new interesting properties emerge in our setting. Also, case (iv) is the multi-agent counterpart of the literature dealing with interior binding types. It has been studied in Brocas (2011).

The situations analyzed in that literature always exhibit an equilibrium utility that is increasing (the standard (IC_1) constraint) and convex (the standard (IC_2) constraint) in the agent's type. Non-monotonic equilibrium utility functions emerge rarely in the contract theory literature.[16] Case (i) departs from the existing literature as it combines both features: reservation utilities are type-dependent and the equilibrium utility is not monotonic. For the case of our allocation mechanism, we will show that our problem can still be solved through standard methods to obtain novel properties.

We will now derive the properties of the optimal allocation mechanisms in cases (i) and (iii). Recall that case (i) can be interpreted as the allocation problem of a drastic innovation, wile case (iii) corresponds rather to the allocation of a minor innovation.

3 Optimal mechanism when $\alpha_a > 0$

Given $\alpha_a > 0$, incentive-compatibility requires that the utility of agent i be convex in v_i ((IC_2)) but not necessarily monotonic ((IC_1)). Also, the reservation utility $w_i(v_i)$ is decreasing in v_i and, the informational rents $\Phi_i(v_i)$ are increasing in v_i. Then, (IR) binds at $\underline{v}$:

$$u_i(\underline{v}) = w_i(\underline{v}) \quad \text{and} \quad u_i(v_i) > w_i(v_i) \quad \forall\, v_i > \underline{v}. \tag{5}$$

Note that for each mechanism A satisfying (IC_2)–(F_0)–(F_1), the convexity of the equilibrium utility implies that there exists at most one valuation $\tilde{v}(A)$ such that[17]:

$$\frac{\mathrm{d}}{\mathrm{d}v_i} u_i(\tilde{v}(A)) = 0. \tag{\underline{U}}$$

[16] See Parlane (2001) for an example. The author studies the optimal allocation of two tasks to two agents with private information. If one task is more valuable than the other, the model can be reinterpreted as a competition between agents, where the "winner" enjoys the valuable task and the "loser" suffers the (type-dependent) externality of getting the least valuable task. In Parlane (2001), the reservation utility is normalized to zero and countervailing incentives arise in the "specialization case" because the difference between valuation and externality is positive for high-type agents and negative for low-type ones. As a result, the total equilibrium utility (which is equal to the informational rents) can be non monotonic. This is also reminiscent of Chen and Potipiti (2010) for the case of allocation mechanisms with positive externalities.

[17] For some mechanisms A, it may a priori be the case that $\mathrm{d}u_i(v_i)/\mathrm{d}v_i < 0$ for all v_i (in which case $\tilde{v}(A) \equiv \overline{v}$) or that $\mathrm{d}u_i(v_i)/\mathrm{d}v_i > 0$ for all v_i (in which case $\tilde{v}(A) \equiv \underline{v}$).

so that agent i's equilibrium utility is decreasing in v_i for all $v_i < \tilde{v}(A)$ and increasing in v_i for all $v_i > \tilde{v}(A)$. Given (5) and using (IC_1) and the integration by parts technique, the seller's optimization program $\mathcal{P}$ is equivalent to program $\mathcal{P}^*$:

$$\mathcal{P}^* : \max \int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{\overline{v}} \Big[X_i(v)\pi_i^*(v) + X_j(v)\pi_j^*(v)\Big] \mathrm{d}F(v_i)\mathrm{d}F(v_j) - 2w_i(\underline{v})$$
$$\text{s.t.} \quad (\text{IC}_2) - (\text{F}_0) - (\text{F}_1),$$

where $\pi_i^* = v_i - \alpha_j(v) - \frac{1-F(v_i)}{f(v_i)} + \alpha_a \frac{1-F(v_j)}{f(v_j)}$ and $\pi_j^* = v_j - \alpha_i(v) - \frac{1-F(v_j)}{f(v_j)} + \alpha_a \frac{1-F(v_i)}{f(v_i)}$.

$\pi_i^*(v)$ is agent i's *virtual surplus* and it represents the net surplus that the auctioneer can extract by selling the good to i rather than keeping it adjusted for the informational rents that she is obliged to grant due to the asymmetry of information with both bidders. Note that the net surplus of selling the good to agent i is $\pi_i^F(v)$. Under asymmetric information, the seller leaves informational rents to both bidders to induce truthful revelation of their valuations. This is reflected in the two distortions $\frac{1-F(v_i)}{f(v_i)}$ and $-\alpha_a \frac{1-F(v_j)}{f(v_j)}$. Increasing the rent of agent i with type v_i requires to increase the rent of any type above v_i, which are in proportion $1 - F(v_i)$. At this stage, we need to introduce the following technical assumption.

Assumption 2a (i) $\frac{\mathrm{d}}{\mathrm{d}v_i}\left[v_i - \frac{1-F(v_i)}{f(v_i)}\right] > \max\{0, \alpha_b\}$ for all v_i and (ii) $\alpha_b \leq 1$.

In auctions with no or fixed externalities ($\alpha_a = \alpha_b = 0$), the monotone hazard rate property ensures that the version of (IC_2) obtained for $\alpha_a = 0$ is satisfied (for free) at equilibrium: it makes the problem regular. In auctions with type-dependent externalities, we need a further condition to make the problem regular. Assumption 2a(i) is sufficient. Condition (ii) guarantees that the allocations under complete and asymmetric information are comparable.[18]

Assumption 3 We consider two cases: (i) $\alpha_a \in (0, 1)$ and $\alpha_b < 0$; (ii) $\alpha_a \geqslant 1$ and $\alpha_b \in (0, 1)$.

These conditions ensure that the reserve price functions in Proposition 1 cross only once.[19]

Proposition 1 *The optimal mechanism A^* solves $\mathcal{P}^*$. It has the following properties:*

$$X_i^*(v_i, v_j) = \begin{cases} 1 & \text{if} \quad v_i > v_j \quad \text{and} \quad v_i > r_i^*(v_j) \\ 0 & \text{otherwise,} \end{cases}$$

[18] The assumption is sufficient. Results (i)–(iii) in Proposition 1 are true under Assumption 2a(i), but point (iv) requires Assumption 2a(ii).

[19] Even though they simplify the proof, they are not critical to determine the qualitative properties of the mechanism.

where $r_i^*(v_j)$ *is the value of* v_i *such that* $\pi_i^*(\tilde{r}_i(v_j), v_j) = 0$. *The mechanism implies that:*

(i) *The agent with lowest valuation never obtains the good* ($v_i > v_j \Leftrightarrow X_j^*(v_i, v_j) = 0$).
(ii) *The reserve price for agent i is increasing in the valuation of agent j* ($\partial r_i^* / \partial v_j > 0$).
(iii) *The type that obtains the minimum equilibrium payoff is* $\tilde{v}(A^*) \in (\underline{v}, \overline{v})$.
(iv) *The good may be allocated more or less often than under full information.*

Proof See Appendix A-3.

The optimal allocation when valuations are privately known is obtained by comparing for each announced pair of valuations the three virtual surplus $\{\pi_1^*, \pi_2^*, 0\}$. First, the probability of obtaining the good is increasing in the agent's type [part (i)]. Intuitively, if $v_i > v_j$, then obviously i benefits more from obtaining the good than j. Given that $\alpha_a > 0$, j's willingness to pay to prevent a sale to i is lower than i's willingness to pay to prevent a sale to j. As a result, the seller always prefers to sell the good to i rather than extract a payment from j and keep the good. So, in equilibrium, either the good is not sold or it is allocated to i. Second, reserve prices are type-dependent and in particular each agent's reserve price is increasing in the valuation of the rival [part (ii)]. As the valuation of the rival increases, the externality the rival may suffer also increases and so does his willingness to pay to avoid a sale. It therefore becomes relatively more profitable to keep the item in exchange of a payment from the rival rather than to sell to the agent. Third, a high-type agent is willing to pay more than his low-type opponent both to get the good and to avoid suffering the externality. As a result, a truthful revelation mechanism implies that informational rents must be increasing in the agent's valuation. However, and contrary to previous analyses, a medium-type agent obtains a smaller equilibrium utility than a low-type or a high-type agent. Indeed, allocating the item to a medium-type or a low-type agent is not very attractive. However, a medium-type agent is willing to pay more to avoid the sale to the rival than a low-type agent. Hence, a low-type agent rarely obtains the good and is extracted few payments, while a medium-type agent obtains rarely the item but is extracted payments in exchange of not selling to his rival. Hence, the equilibrium utility of a medium-type agent is smaller than that of a low-type agent [part (iii)]. This is a consequence of Lemma 2. Fourth, in order to decrease informational rents, and compared to the full information case, the good must be allocated less often when agents have relatively low valuations and more often when agents have relatively high valuations [part (iv)]. Hence, on average, the good is sold more or less often than under full information depending on the exact shape of the externality. This result contrasts with the standard auction mechanisms, including auctions with (non type-dependent) externalities, in which the trade-off between rents and efficiency is always solved by reducing the likelihood of allocating the item. The equilibrium allocation is depicted in Fig. 2.

Last, it is easy to see why the problem is regular under Assumption 2a. Under the assumption, the reserve price faced by i is increasing in the valuation of j. When v_i increases, i is more likely to exceed the reserve prices $r_i^*(v_j)$ and $E_{v_j} X_i(v)$ increases.

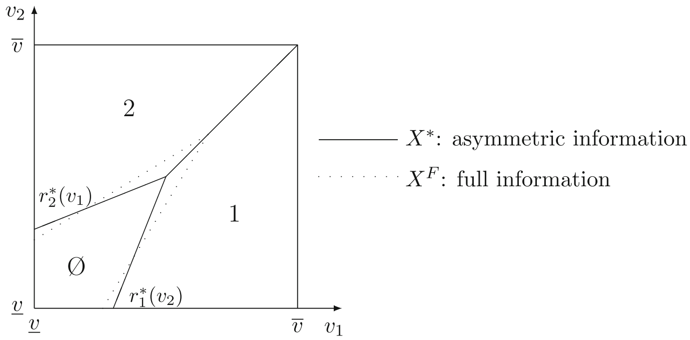

Fig. 2 Optimal allocation when $\alpha_a > 0$

Moreover, j is less likely to have a valuation above $r_j^*(v_i)$, and $E_{v_j} X_j(v)$ decreases. This delivers (IC_2) for free.

Remark 1 When $\alpha_a = 0$, then (IR) binds at $\underline{v}$ (see Fig. 1c). If we further replace $\alpha_j(v_i) = \alpha_b v_i + \gamma$ by α^j (i.e. each agent suffers a different externality but it is fixed and uncorrelated with his type), then we are exactly in the case analyzed by Jehiel et al. (1996).

4 Optimal mechanism when $\alpha_a \leq -1$

When $\alpha_a \leq -1$, incentive-compatibility requires that the utility of agent i be convex in v_i (see (IC_2)) and monotonic (see (IC_1)). At the same time, the reservation utility $w_i(v_i)$ is linearly increasing in v_i and the informational rents $\Phi_i(v_i)$ are decreasing in v_i. At equilibrium, (IR) will bind at $\overline{v}$:

$$u_i(\overline{v}) = w_i(\overline{v}) \quad \text{and} \quad u_i(v_i) > w_i(v_i) \forall\, v_i < \overline{v}. \tag{6}$$

Given (6) and using (IC_1) and the integration by parts technique, the seller's optimization program $\mathcal{P}$ is now equivalent to program $\mathcal{P}^{**}$:

$$\begin{aligned} \mathcal{P}^{**} \; : \; \max & \int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{\overline{v}} \Big[X_i(v)\pi_i^{**}(v) + X_j(v)\pi_j^{**}(v) \Big] \mathrm{d}F(v_i)\mathrm{d}F(v_j) - 2w_i(\overline{v}) \\ \text{s.t.} \quad & (\mathrm{IC}_2) - (\mathrm{F}_0) - (\mathrm{F}_1), \end{aligned}$$

where $\pi_i^{**}(v) = v_i - \alpha_j(v) + \frac{F(v_i)}{f(v_i)} - \alpha_a \frac{F(v_j)}{f(v_j)}$ and $\pi_j^{**}(v) = v_j - \alpha_i(v) + \frac{F(v_j)}{f(v_j)} - \alpha_a \frac{F(v_i)}{f(v_i)}$.

The interpretation of the virtual surplus is the same as before, except that the distortion due to informational rents act differently. Here, increasing the rent of agent i with type v_i requires to increase the rent to any agent with type below v_i, which are in proportion $F(v_i)$.

Assumption 2b $\frac{d}{dv_i}\left[v_i + \frac{F(v_i)}{f(v_i)}\right] > \alpha_b \geqslant 0$ for all v_i.

It is the counterpart of Assumption 2a.

Assumption 4 We consider two cases: (i) $\alpha_a < -1$ and $\alpha_b \geqslant 0$; (ii) $\alpha_a = -1$ and $\alpha_b > 0$.

These technical conditions ensure that the the seller is not indifferent between allocating the good to agent i or agent j when valuations differ.

Proposition 2 *The optimal mechanism A^{**} solves $\mathcal{P}^{**}$. It has the following properties:*

$$X_i^{**}(v_i, v_j) = \begin{cases} 1 & \text{if } v_i < v_j \text{ and } v_i > r_i^{**}(v_j) \\ 0 & \text{otherwise,} \end{cases}$$

where $r_i^{**}(v_j)$ is the value of v_i such that $\pi_i^{**}(r_i^{**}(v_j), v_j) = 0$.[20]

The mechanism implies that:

(i) *The agent with highest valuation never obtains the good ($v_i > v_j \Leftrightarrow X_i^{**}(v_i, v_j) = 0$).*
(ii) *The reserve price for agent i is decreasing in the valuation of agent j ($\partial r_i^{**}/\partial v_j < 0$).*
(iii) *The good is allocated more often than under full information.*

Proof See Appendix A-4.

The properties of the mechanism are as follows. First, the good is never allocated to the agent with highest valuation among the two. Recall that agents are willing to pay not only to obtain the good but also to prevent the sale to the rival. When $\alpha_a \geqslant -1$, low-type agents have more to lose if the rival obtains the good than high type agents. They are therefore willing to pay to avoid the sale. Formally, if $v_i > v_j$, i benefits more from obtaining the good than j. However, given that $\alpha_a \leq -1$, then j's willingness to pay to prevent a sale to i is much higher than i's willingness to pay to prevent a sale to j. As a result, the seller always prefers to extract a payment from j and keep the good rather than sell it to i. So, in equilibrium, either the good is not sold or it is allocated to j.[21] This feature of the mechanism does not emerge in other studies of auctions with negative externalities. The articles focusing on externalities that are not type-dependent cannot deliver such result: type-dependency is a necessary condition for our result. Moreover, the result holds when the externality satisfies certain conditions: low-type agents must be willing to pay enough to avoid the sale to high-type agents so that the seller find it profitable to not allocate the item to high-type agents.

[20] Stated differently, $X_i^{**}(v) = 1$ iff $\pi_i^{**}(v) > \max\{0, \pi_j^{**}(v)\}$ and $X_j^{**}(v) = 1$ iff $\pi_j^{**}(v) > \max\{0, \pi_i^{**}(v)\}$. Note that if $\pi_i^{**}(v_i, v_j) > 0$ for all v_i then $r_i^{**}(v_j) \equiv \underline{v}$ and if $\pi_i^{**}(v_i, v_j) < 0$ for all v_i then $r_i^{**}(v_j) \equiv \overline{v}$.

[21] This is reminiscent of the complete information case: given $1 + \alpha_a \leq 0$ and Assumption 2a, we have $1 + \alpha_a \leq \alpha_b$ and it is optimal to allocate the good to the agent with the smallest valuation.

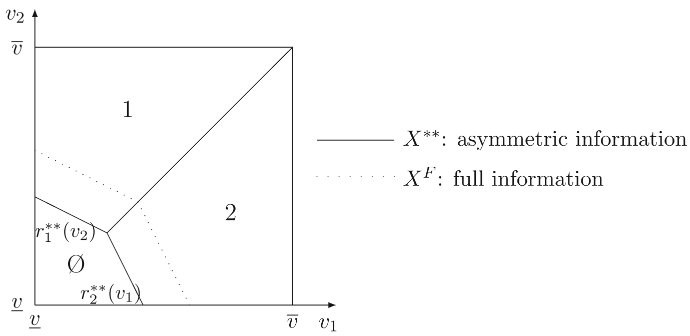

Fig. 3 Optimal allocation when $\alpha_a < -1$

This type of situation has not been studied earlier. We shall see in Sect. 5 that our result has other related interesting consequences. Second, the reserve price faced by each agent is now decreasing in the valuation of his competitor. The idea is simply that, as v_i increases, the externality suffered by i if the good is sold to j decreases, and so is i's willingness to pay to avoid that sale. Therefore, it becomes relatively more beneficial to sell the good to j. This is reflected in a lower reserve price r_j^{**} [part (ii)]. Last, the good is allocated more often than under full information. As in all the contracting literature, the auctioneer solves the standard trade-off efficiency vs. rents. In traditional auction models, in order to limit the informational rents, the auctioneer decreases the likelihood of selling the good. However, in our case, the mechanism to diminish the informational rents is precisely the opposite, that is to increase the likelihood of selling the good ($X_i^{**} \geqslant X_i^F$) [part (iv)].[22] By doing so, externalities are exerted also more often than in the first best scenario. These results are represented in Fig. 3.[23]

5 Second price sealed bid auctions

The two previous sections showed that the properties of the optimal mechanism are tied to the specific nature of the externalities. In terms of the examples, this means in particular that the allocation of licenses should exhibit different rules depending on the exact nature of the technology. We believe this is important as the earlier literature on auctions with externalities focuses on this type of good. Of course, the argument applies to other settings in which agents in the auction are engaged in ex-post interactions (e.g., competitors in lumber auctions, procurement contracts auctions, etc).

[22] To see this, note that when (IR) binds at the top, informational rents are smaller the steeper the slope of the equilibrium utility. Rents decrease when X_i^{**} and X_j^{**} increase.

[23] Again, it is easy to see why the problem is regular under Assumption 2b. Indeed, the reserve price faced by agent i is decreasing in the valuation of agent j. When v_i increases, other things being equal, i is more likely to exceed the reserve prices $r_i^{**}(v_j)$ and j is also more likely to have a valuation above $r_j^{**}(v_i)$. Then, both $E_{v_j} X_i(v)$ and $E_{v_j} X_j(v)$ increase in v_i, which delivers (IC$_2$) for free.

To take this prediction a step further, we now ask the question of how the mechanism could be implemented through auction procedures. We start by analyzing the allocation of the item via a pure second price sealed bid auction, and we then determine how such procedure should be modified to implement the optimal mechanism outcome. Let us denote by b_i the bid of agent i. Our first result is as follows:

Proposition 3 *In a second price sealed bid auction, the equilibrium pure symmetric bidding strategy is* $b(v_i) = v_i + \alpha_i(v_i, v_i)$. *When* $1 + \alpha_a > 0$, *it is increasing in* v_i *and it exists if* $1 + \alpha_a + \alpha_b \geqslant 0$. *When* $1 + \alpha_a < 0$, *it is decreasing in* v_i *and it exists if* $1 + \alpha_a + \alpha_b \leq 0$.

Proof See Appendix A-5.

When $\alpha_a \leq -1$, then the agents with highest valuations will always submit the lowest bids. This constitutes a novelty in the literature of auctions with private values.[24] The idea is simply that if agent i has a higher valuation than agent j, then j has a much higher willingness to pay than i to prevent the auctioneer from selling the good to the rival. As a result, j is willing to submit a higher bid than i, because obtaining the good is the best insurance against suffering the externality.

Each agent bids for two motives. First, i is ready to pay up to v_i to obtain the good. Second, i is willing to pay up to $\alpha_i(v_i, v_j)$ to prevent a rival with type v_j from getting the good (an insurance). In equilibrium, each agent bids the sum of his valuation and the externality he suffers if his rival is exactly his type. If j plays that strategy, i wins at the correct time if he bids $v_i + \alpha(v_i, v_i)$. Deviating results in losing or winning at too high a price. It is interesting to note that the strategy is not dominant when $\alpha_b \neq 0$. This is the case because i needs to know the valuation of agent j to determine the externality he will suffer if j wins. In the absence of externalities, the bidding strategy is dominant because i cares only about the bid of his rival and not about his valuation. This is also true as long as the externality is fixed or when $\alpha_b = 0$. Last, there is separation only if a further regularity condition is satisfied. This is reminiscent of Jehiel and Moldovanu Jehiel and Moldovanu (2000).[25] Interestingly, it is also reminiscent of Milgrom and Weber Milgrom and Weber (1982). To see this, note that there are only two outcomes for agent i in a pure second price sealed bid auction: either he obtains the good or his rival obtains the good. Therefore, the willingness to pay of agent i corresponds to his willingness to pay to get the good and avoid the sale to his competitor at the same time (in which case the first outcome realizes). It is then equal to the *modified value* $v_i + \alpha_i(v_i, v_j)$. Given type-dependency, values are interdependent as in Milgrom and Weber Milgrom and Weber (1982).[26]

[24] This feature can arise in other settings. See Moldovanu and Sela (2003) for a case with interdependent values.

[25] The case studied in Jehiel and Moldovanu Jehiel and Moldovanu (2000) is qualitatively equivalent to the situation where $1 + \alpha_a \geqslant 0$ and $1 + \alpha_a + \alpha_b \geqslant 0$. The authors derive the same bidding strategies, which always exist in their setting. Also, given their assumptions, the optimal bidding strategy cannot be decreasing.

[26] In the case the pure second price sealed bid auction, our model is equivalent to the model of interdependent values with independent signals. To see this, when $1 + \alpha_a > 0$, the optimal symmetric increasing

Remark 2 We have concentrated on auctions that maximize revenue. The reader might wonder what an efficient auction looks like in our setting. Efficiency requires to allocate the good to i when $v_i - \alpha_j(v) > v_j - \alpha_i(v)$. In other words, efficiency corresponds to the first best scenario under revenue maximization. This implies in particular that it is efficient to allocate the good to the agent with the highest valuation when $1+\alpha_a > \alpha_b$, and to the agent with the lowest valuation when $1+\alpha_a < \alpha_b$.[27] Given Proposition 3 and the above conditions, the second price sealed bid auction is efficient when $1+\alpha_a \geq \max\{0, \alpha_b\}$ and when $1+\alpha_a \leq \min\{0, \alpha_b\}$.

In what follows, we show how mechanisms A^* and A^{**} can be implemented with a suitably modified second price sealed bid auction.

Proposition 4 *The mechanisms* (A^*, A^{**}) *can be implemented with a modified second price sealed bid auction in which agent i pays an entry fee c, faces a reserve price contingent on the bid of the rival* $R_i(b_j)$*, and plays a pure bidding strategy. Provided the bidding strategy exists,*

- *In mechanism* A^{**}*, i is allocated the good if his bid is smaller than* $R_i^{**}(b_j)$ *where* $R_i^{**}(b_j)$ *decreases with the rival's bid* $(\partial R_i^{**}/\partial b_j < 0)$*. The agent's optimal bidding strategy* $b^{**}(v_i)$ *decreases with his valuation* $(\partial b^{**}/\partial v_i < 0)$*. Besides,* $R_i^{**}(b^{**}(v_j)) \equiv b^{**}(r_i^{**}(v_j))$ *where* $r_i^{**}(v_j)$ *is the optimal reserve price in* A^{**}*.*
- *In mechanism* A^**, i is allocated the good if his bid is higher than* $R_i^*(b_j)$ *where* $R_i^*(b_j)$ *increases with the rival's bid* $(\partial R_i^*/\partial b_j > 0)$*. The agent's optimal bidding strategy* $b^*(v_i)$ *increases with his valuation* $(\partial b^*/\partial v_i > 0)$*. Besides,* $R_i^*(b^*(v_j)) \equiv b^*(r_i^*(v_j))$ *where* $r_i^*(v_j)$ *is the optimal reserve price in* A^**.*

Proof See Appendix A-6.

As usual, the seller resorts to an entry fee in order to extract maximum payments from the bidders. Agents are willing to incur this cost because they are threatened to suffer the externality for sure if they refuse to participate. The implementation has three main novelties with respect to the cases already analyzed in the literature.

First, the reserve price of each agent depends on the bid of his competitor. This is a consequence of the reserve price in the optimal mechanism being a function of the competitor's valuation (see Propositions 1 and 2). Second, mechanism A^* can only be

Footnote 26 continued
bidding strategy maximizes $\int_{\underline{v}}^{b^{-1}(b_i)}(v_i - b(s))f(s)ds - \int_{b^{-1}(b_i)}^{\overline{v}} \alpha_i(v_i, s)f(s)ds \equiv \int_{\underline{v}}^{b^{-1}(b_i)}(v_i + \alpha_i(v_i, s) - b(s))f(s)ds - \int_{\underline{v}}^{\overline{v}} \alpha_i(v_i, s)f(s)ds$. Given the second term is independent of b_i, the maximization of this function is equivalent to the maximization of $\int_{\underline{v}}^{b^{-1}(b_i)}(z_i(v_i, s) - b(s))f(s)ds$ where $z(v_i, v_j) = v_i + \alpha_i(v_i, v_j)$ is i's valuation. Note that this is also reminiscent to the case with private values when the distribution is symmetric in Dasgupta and Maskin (2000).

[27] Note also that, at the efficient solution, the good is allocated to i when his modified value $v_i + \alpha_i(v_i, v_j)$ exceeds j's modified value $v_j + \alpha_j(v_j, v_i)$. The condition $1+\alpha_a > \alpha_b$ means that buyer i's valuation has a greater marginal effect on i's modified value than on the modified value of buyer j. This condition is equivalent to the condition that is necessary to get efficiency in Dasgupta and Maskin (2000). In our setting, it is also sometimes efficient to allocate the good to the agent with the lowest valuation. This is true when $1+\alpha_a < \alpha_b$, that is when buyer i's valuation has a smaller marginal effect on i's modified value than on the modified value of buyer j.

implemented with a sealed bid auction in which the bid of each agent is increasing in his valuation, and mechanism A^{**} can only be implemented with a sealed bid auction in which the bid of each agent is decreasing in his valuation. This is a consequence of Propositions 1, 2, and 3. Third, the optimal reserve price is a price ceiling when $\alpha_a \leq -1$. This is true because it is optimal to sell the good to i if his valuation is high enough ($v_i > r_i^{**}(v_j)$) while bids decrease in valuations.[28] The seller commits herself to not accept high bids in order to extract payments even when the good is not sold. In A^{**}, high bids come from low types who primarily wish to get insured against the externality. If $v_1 > v_2$, it is optimal to extract the part of the payment agent 2 wishes to make to avoid the sale to agent 1. However, given agent 2 values the good itself very little, it is better to not extract the payment he is willing to make to obtain the item and, instead, extract the payment agent 1 wishes to make to avoid the sale to agent 2. Overall, high bids are shut down and entry fees are collected.

The optimal mechanisms A^* and A^{**} can be implemented with the modified second price sealed bid auction if the required bidding strategies exist. It turns out that the bidding strategies are difficult to characterize analytically for general distributions. We provide a full characterization of the equilibrium in the case of a uniform distribution on (0, 1) and when $\alpha_b = 0$ in Appendix B. A full implementation requires to resort to additional taxes/subsidies in that case. We illustrate it with the following numerical example. Suppose $\alpha_a = -2$ and $\gamma = 4$, the optimal reserve price is $r_i^{**} = 1$ if $v_i < 1/2$ and $-2v_i + 2$ when $v_i \geqslant 1/2$. The price ceiling $R_j^{**}(b_i)$ and the pure bidding strategy $b_i^{**}(v_i)$ are

$$b_i^{**}(v_i) = \begin{cases} 1 & v_i < 1/2 \\ -2v_i + 2 & v_i \in (1/2, 2/3) \\ -5v_i + 4 & v_i \in (2/3, 4/5) \\ 0 & v_i > 4/5 \end{cases},$$

$$R_j^{**}(b_i) = \begin{cases} 1 & b_i < 1/2 \\ -4/5b_i + 6/5 & b_i \in (1/2, 2/3) \\ -5b_i + 4 & b_i \in (2/3, 4/5) \\ 0 & b_i > 4/5. \end{cases}$$

Note that the bidding strategy in the pure second price sealed bid auction would be $v_i + (-2v_i + 4)$. Adding a fixed price ceiling results in truncating the bidding function: some valuations bid according to the same bidding strategy and some others bid at the price ceiling. When the price ceiling is bid dependent, the bidding strategy itself is affected. By placing a bid b_i, not only agent i affects his own chances of winning but also the "standards" faced by his rival. When b_i increases, the price ceiling faced by j decreases by -2 resulting in a lower chance of suffering the externality. Combining both effects, low valuations bid the reserve price $r_i^{**}(v_i)$ and high valuations bid an amount reflecting the basic bid in a second price sealed bid auction $v_i + (-2v_i + 4)$ and the added value of affecting the price ceiling of the rival $+2(-2v_i + 4)$. Furthermore, to implement the optimal mechanism, it is necessary to add a tax/subsidy on some

[28] Note that this is also true under complete information.

bids. In the example, the seller taxes all winning bids below 2/3 by an amount of 8. This distort further the strategy of high valuation bidders. Overall, their equilibrium bidding strategy is $b_i^{**}(v_i) = v_i + (-2v_i + 4) + 2(-2v_i + 4) - 8 = -5v_i + 4$.[29] Last, the entry fee in this knife-edge example is $c = 0$. This is the case because the binding type ($v_i = 1$) always suffers the externality in the second price sealed bid auction. There is no extra payment to collect from him.

6 Conclusion

This paper has extended previous works of auctions with type-dependent negative externalities. The analysis of the allocation mechanism has different properties depending on whether externalities are strongly decreasing or increasing in the agent's valuation, leading to two novel theoretical features. First, when externalities are increasing in the valuation, the equilibrium utility of an agent is non monotonic in the valuation. Typically, there exists an interior type that obtains the lowest level of utility. Second, when externalities are strongly decreasing in the valuation, the good might be sold to the agent with lowest valuation and the seller may allocate the good with higher probability than in the full information case. If the seller uses a second price sealed bid auction, the bids will be decreasing in valuations, and she should resort to a bid-dependent price ceiling to implement the optimal auction. Last, from an applied perspective, we have shown that modeling externalities is critical as different models lead to different predictions in terms of mechanism design.

We would like to conclude by pointing out a few alleys for future research. First, it is natural to extend the analysis to the case where valuations are also correlated. Second, the current analysis abstracts from the fact that the outcome of the auction reveals information to participants. One way to justify this is to assume that information is revealed ex-post. In the case where ex-post payoffs are supposed to depend on non observable ex-post types, there is a possible informational leakage in the auction. Of course, this must be anticipated at the time of placing bids.[30] Third, it could be also interesting to analyze the case of positive externalities and to determine whether some of our results obtain there as well. For instance, in cases in which higher types enjoy a relatively bigger externality than lower types, the latter should be granted the good and the mechanism should also be implemented with a price ceiling. Last, we have seen that in the presence of externalities, each player has two distinct valuations. One to obtain the good and a second to avoid the sale to his competitor. We have analyzed a case in which both are linked and can be inferred from each other. In other settings, the two valuations may not be fully correlated. In that case, asking to send a unique message on the willingness to pay restricts the ability to elicit the two distinct valuations. It would be interesting to characterize auction procedures that implement the

[29] This ensures that the bidding strategy is strictly increasing at the time required by the optimal mechanism. See Appendix for details.

[30] This issue has been studied in Goeree (2003) for the case where only the type of the winner plays a role in the aftermarket and for specific auction procedures.

optimal auction. We conjecture that agents should be asked to submit a bid to obtain the good as well as a bid to avoid the sale to the competitor.

Appendix A

Appendix 1

Note that $u_i(v_i, v_i') = u_i(v_i', v_i') + (v_i - v_i')[E_{v_j} X_i(v_i', v_j) - \alpha_a E_{v_j} X_j(v_i', v_j)]$. Then the incentive-compatibility constraint is equivalent to:

$$u_i(v_i, v_i) \geq u_i(v_i', v_i') + (v_i - v_i')[E_{v_j} X_i(v_i', v_j) - \alpha_a E_{v_j} X_j(v_i', v_j)]. \tag{7}$$

Using this inequality twice, the incentive-compatibility constraint is equivalent to

$$\begin{aligned} &(v_i - v_i')[E_{v_j} X_i(v_i', v_j) - \alpha_a E_{v_j} X_j(v_i', v_j)] &(8)\\ &\quad \leq u_i(v_i, v_i) - u_i(v_i', v_i') \\ &\quad \leq (v_i - v_i')[E_{v_j} X_i(v_i, v_j) - \alpha_a E_{v_j} X_j(v_i, v_j)]. &(9)\end{aligned}$$

Then the agent reveals truthfully if:

$$E_{v_j}\left[X_i(v_i', v_j) - \alpha_a X_j(v_i', v_j)\right] \leq E_{v_j}\left[X_i(v_i, v_j) - \alpha_a X_j(v_i, v_j)\right] \quad \forall\, v_i' \leq v_i. \tag{IC$_2$}$$

(8) must hold for all v_i' and all $v_i = v_i' + \delta$ with $\delta > 0$. Since $E_{v_j} X_i(v_i, v_j) - \alpha_a E_{v_j} X_j(v_i, v_j)$ is increasing in v_i, we can take the Riemann integral. Then, the agent reveals truthfully if we also have:

$$u_i(v_i) - u_i(v_i') = \int_{v_i'}^{v_i} E_{v_j}[X_i(s, v_j) - \alpha_a X_j(s, v_j)]\mathrm{d}s \quad \forall\, v_i' \leq v_i. \tag{IC$_1$}$$

To complete the proof, we need to verify that (IC$_1$) and (IC$_2$) imply (7). Suppose $v_i' \leq v_i$, then given (IC$_1$) and (IC$_2$), we have:

$$\begin{aligned} u_i(v_i, v_i) &= u_i(v_i', v_i') + \int_{v_i'}^{v_i} E_{v_j}[X_i(s, v_j) - \alpha_a X_j(s, v_j)]\mathrm{d}s \\ &\geq u_i(v_i', v_i') + \int_{v_i'}^{v_i} E_{v_j}[X_i(v_i', v_j) - \alpha_a X_j(v_i', v_j)]\mathrm{d}s \\ &= u_i(v_i', v_i') + (v_i - v_i')[E_{v_j} X_i(v_i', v_j) - \alpha_a E_{v_j} X_j(v_i', v_j)]. \end{aligned}$$

The seller maximizes her expected revenue (the sum of transfers) under constraints (IC_1) and (IC_2) (to induce truth-telling) and the remaining constraints (IR), (F_0) and (F_1).[31] □

Appendix 2

The proof is obtained by inspection of (3) and (4).

When $\alpha_a > 0$, we can only show that $\frac{d}{dv_i} u_i(v_i) = E_{v_j}[X_i(v)] - \alpha_a E_{v_j}[X_j(v)] < E_{v_j}[X_i(v)]$, and therefore $u_i(v_i)$ may not be monotonic. However, $\frac{\mathrm{d}}{\mathrm{d}v_i}\Phi_i(v_i) = E_{v_j}[X_i(v)] + \alpha_a(1 - E_{v_j}[X_j(v)]) > 0$ which proves that the rent is increasing. The binding type is $\underline{v}$.

When $\alpha \leq -1$, we have $\frac{\mathrm{d}}{\mathrm{d}v_i} u_i(v_i) = E_{v_j}[X_i(v)] - \alpha_a E_{v_j}[X_j(v)] > 0$ and $\frac{\mathrm{d}}{\mathrm{d}v_i}\Phi_i(v_i) = E_{v_j}[X_i(v)] + \alpha_a(1 - E_{v_j}[X_j(v)]) < 0$. The rent is decreasing and the binding type is $\overline{v}$.

When $\alpha \in (-1, 0)$, we have $\frac{\mathrm{d}}{\mathrm{d}v_i} u_i(v_i) = E_{v_j}[X_i(v)] - \alpha_a E_{v_j}[X_j(v)] > 0$ but $\frac{d}{dv_i}\Phi_i(v_i) \gtrless 0$. The rent may not be monotonic and the binding type may be interior.

To economize notations, let $H(v_i) = E_{v_j}[X_i(v_i, v_j) - \alpha_a X_j(v_i, v_j)]$ from now on.

Appendix 3

The mechanism such that the seller keeps the good if $\max_i\{\pi_i^*(v)\} < 0$ and allocates it to the bidder with the highest $\pi_i^*(v)$ otherwise maximizes $\mathcal{P}^*$ under (F_0) and (F_1). We have $\pi_i^*(v) > \pi_j^*(v)$ if $v_i > v_j$. Moreover, when the cutoff $r_i^*(v_j)$ is interior, we have

$$\left[\frac{\mathrm{d}}{\mathrm{d}v_i}\left[v_i - \frac{1 - F(v_i)}{f(v_i)}\right]|_{r_i^*} - \alpha_b\right]\frac{\mathrm{d}}{\mathrm{d}v_j} r_i^*(v_j) - \alpha_a \frac{\mathrm{d}}{\mathrm{d}v_j}\left[v_j - \frac{1 - F(v_j)}{f(v_j)}\right] = 0$$

proving that $r_i^*(v_j)$ is increasing in v_j under Assumption 2a(i). We now show that (IC_2) is satisfied.

Curves $r_i^*(v_j)$ and $r_j^{*^{-1}}(v_j)$ cross at $\breve{v}$. $\frac{\partial}{\partial v_j} r_i^*(v_j)|_{v^o} \leq 1$ when $\alpha_a < 1$ and $\alpha_b < 0$, and $\frac{\partial}{\partial v_j} r_i^*(v_j)|_{\breve{v}} \geq 1$ when $\alpha_a > 1$ and $\alpha_b > 0$. In both cases $\breve{v}$ is unique.[32] We work under that assumption (Assumption 3 in the text).

Case 1: When $\alpha_a < 1$ and $\alpha_b < 0$, $r_j^*(v_i) \gtrless r_i^{*^{-1}}(v_i)$ for all $v_i \lessgtr \breve{v}$. There also exists v' such that $r_i^{*^{-1}}(v') = \underline{v}$. When $v_i < v'$, $E_{v_j} X_i(v) = 0$ and $E_{v_j} X_j = 1 - F(r_j^*(v_i))$. When $v_i \in [v', \breve{v}]$, $E_{v_j} X_i(v) = F(r_i^{*^{-1}}(v_i))$ and $E_{v_j} X_j = 1 - F(r_j^*(v_i))$. When $v_i > \breve{v}$, then $E_{v_j} X_i(v) = F(v_i)$ and $E_{v_j} X_j(v) = 1 - F(v_i)$. Then ($IC_2$) is

[31] Note that the proof is similar to Myerson (1981) except that we do not provide a sufficient condition for (IR) to hold at this stage.

[32] Note that $\frac{\partial}{\partial v_j} r_i^*(v_j)|_{\breve{v}} = \frac{\alpha_a h'}{h' - \alpha_b}$ where $h' = \frac{\partial}{\partial v_i}[v_i - \frac{1-F(v_i)}{f(v_i)}]|_{\breve{v}}$.

satisfied everywhere. Last $\tilde{v}(A^*) < \overline{v}$ because $H(\overline{v}) = 1$, and $\tilde{v}(A^*) > \underline{v}$ if $H(\underline{v}) = -\alpha_a(1 - F(r_j^*(\underline{v})) < 0$, that is if $r_j^*(\underline{v}) \neq \overline{v}$.

Case 2: When $\alpha_a > 1$ and $\alpha_b > 0$, $r_j^*(v_i) \lesseqgtr r_i^{*^{-1}}(v_i)$ for all $v_i \lesseqgtr \breve{v}$. There exists $v' > \breve{v}$ such that $r_j^*(v') = \overline{v}$. When $v_i < \breve{v}$, $E_{v_j} X_i(v) = F(v_i)$ and $E_{v_j} X_j = 1 - F(v_i)$. When $v_i \in (\breve{v}, v')$, $E_{v_j} X_i(v) = F(r_i^{*^{-1}}(v_i))$ and $E_{v_j} X_j = 1 - F(r_j^*(v_i))$. When $v_i > v'$, then $E_{v_j} X_i(v) = F(r_i^{*^{-1}}(v_i))$ and $E_{v_j} X_j(v) = 0$. Again (IC$_2$) is satisfied everywhere. Last $\tilde{v}(A^*) > \underline{v}$ because $H(\underline{v} - \alpha_a < 0$ and $\tilde{v}(A^*) < \overline{v}$ if $H(\overline{v}) = F(r_i^{*^{-1}}(\overline{v})) > 0$, that is if $r_i^{*^{-1}}(\overline{v}) \neq \underline{v}$.

Last, let $r_i^F(v_j) = \min\{v_i | \pi_i^F(v_i, v_j) \geqslant 0\}$. For all v_j, there exists $q(v_j)$ such that $\pi_i^*(q(v_j), v_j) = \pi_i^F(q(v_j), v_j)$ and $r_i^*(v_j) \lesseqgtr r_i^F(v_j)$ when $v_i \gtreqless q(v_j)$. □

Appendix 4

The mechanism such that the seller keeps the good if $\max_i\{\pi_i^{**}(v)\} < 0$ and allocates it to the bidder with the highest $\pi_i^{**}(v)$ maximizes $\mathcal{P}^{**}$ under (F$_0$) and (F$_1$). We have $\pi_i^{**}(v) > \pi_j^{**}(v)$ if $v_i < v_j$ (under Assumption 4). When the cutoff is interior,

$$\left[\frac{\mathrm{d}}{\mathrm{d}v_i}\left[v_i + \frac{F(v_i)}{f(v_i)}\right]|_{r_i^{**}} - \alpha_b\right]\frac{\mathrm{d}}{\mathrm{d}v_j} r_i^{**}(v_j) - \alpha_a \frac{\mathrm{d}}{\mathrm{d}v_j}\left[v_j + \frac{F(v_j)}{f(v_j)}\right] = 0$$

showing that $r_i^{**}(v_j)$ is decreasing in v_j under Assumption 2b. We need to check now that (IC$_2$) is satisfied.

Curves $r_i^{**}(v_j)$ and $r_j^{**^{-1}}(v_j)$ cross at $\breve{v}$. We have $\frac{\mathrm{d}}{\mathrm{d}v_i}[v_i + \frac{F(v_i)}{f(v_i)}]|_{\breve{v}} = h$, then $\frac{\partial}{\partial v_j} r_i^{**}(v_j)|_{\breve{v}} = \frac{\alpha_a h}{h - \alpha_b} < -1$, which ensures that $\breve{v}$ is unique. For all $v_i \lesseqgtr \breve{v}$, $r_j^{**}(v_i) \gtreqless r_i^{**^{-1}}(v_i)$. When $v_i < \breve{v}$, $E_{v_j} X_j^{**} = 0$ and $E_{v_j} X_i^{**} = 1 - F(r_i^{**^{-1}}(v_i))$. When $v_i > \breve{v}$, $E_{v_j} X_i^{**} = 1 - F(v_i)$ and $E_{v_j} X_j^{**} = F(v_i) - F(r_j^{**}(v_i))$. Therefore, (IC$_2$) is satisfied everywhere.

Last, $\pi_i^{**}(v) > \pi_i^F(v)$ and therefore $r_i^{**}(v_j) < r_i^F(v_j)$. □

Appendix 5

The proof proceeds in two steps.

Step 1: we first show that, provided it exists, the equilibrium pure symmetric bidding strategy is monotonically increasing or decreasing in v_i as a function α_a.

In a second price sealed bid auction, i's utility is $u_i(v_i, b_i) = v_i - b_j$ if $b_i > b_j$ and $-\alpha_i(v_i, v_j)$ if $b_i < b_j$. Let us denote by $b(v_i)$ the bidding strategy of an agent with valuation v_i. We look for a pure strategy bayesian equilibrium such that $b(\cdot)$ is monotonic. The strategy satisfies:

$$b(v_i) \in \arg\max \int_{\{v_j | b(v_j) < b_i\}} (v_i - b(v_j))\mathrm{d}F(v_j) - \int_{\{v_j | b(v_j) > b_i\}} \alpha_i(v_i, v_j)\mathrm{d}F(v_j).$$

Consider two types v_i' and v_i''. Equilibrium requires that an agent with type v_i' prefers $b(v_i')$ to $b(v_i'')$ and an agent with type v_i'' prefers $b(v_i'')$ to $b(v_i')$. Moreover,

$$u_i(v_i', b(v_i'')) = u_i(v_i'', b(v_i'')) + \int_{\{v_j | b(v_j) < b(v_i'')\}} (v_i' - v_i'')\mathrm{d}F(v_j) - \int_{\{v_j | b(v_j) > b(v_i'')\}} \alpha_a(v_i' - v_i'')\mathrm{d}F(v_j).$$

Then in equilibrium, we have $u_i(v_i', b(v_i')) \geqslant u_i(v_i', b(v_i''))$ which is equivalent to:

$$u_i(v_i', b(v_i')) - u_i(v_i'', b(v_i'')) \geqslant \int_{\{v_j | b(v_j) < b(v_i'')\}} (v_i' - v_i'')dF(v_j) - \int_{\{v_j | b(v_j) > b(v_i'')\}} \alpha_a(v_i' - v_i'')dF(v_j).$$

Similarly, $u_i(v_i'', b(v_i'')) \geqslant u_i(v_i'', b(v_i'))$ implies that:

$$u_i(v_i', b(v_i')) - u_i(v_i'', b(v_i'')) \leq \int_{\{v_j | b(v_j) < b(v_i')\}} (v_i' - v_i'')\mathrm{d}F(v_j) - \int_{\{v_j | b(v_j) > b(v_i')\}} \alpha_a(v_i' - v_i'')\mathrm{d}F(v_j).$$

Overall, equilibrium bids are such that

$$\int_{\{v_j | b(v_j) < b(v_i'')\}} (v_i' - v_i'')\mathrm{d}F(v_j) - \int_{\{v_j | b(v_j) > b(v_i'')\}} \alpha_a(v_i' - v_i'')\mathrm{d}F(v_j) \leq \int_{\{v_j | b(v_j) < b(v_i')\}} (v_i' - v_i'')\mathrm{d}F(v_j) - \int_{\{v_j | b(v_j) > b(v_i')\}} \alpha_a(v_i' - v_i'')\mathrm{d}F(v_j).$$

Suppose $v_i'' < v_i'$, the last inequality implies that

$$\mathrm{Prob}(b(v_j) < b_i'') - \alpha_a \mathrm{Prob}(b(v_j) > b_i'') \leq \mathrm{Prob}(b(v_j) < b_i') - \alpha_a \mathrm{Prob}(b(v_j) > b_i'). \tag{9}$$

The previous equation writes simply as:

$$(1 + \alpha_a)\mathrm{Prob}(b(v_j) < b_i'') \leq (1 + \alpha_a)\mathrm{Prob}(b(v_j) < b_i').$$

Thus, if $1+\alpha_a > 0$, we have $\text{Prob}(b(v_j) < b_i'') \leq \text{Prob}(b(v_j) < b_i')$ and $b_i'' < b_i'$, i.e. $b(v)$ is increasing in v. If $1+\alpha_a < 0$, we have $\text{Prob}(b(v_j) < b_i'') > \text{Prob}(b(v_j) < b_i')$ and $b_i'' > b_i'$, i.e. $b(\cdot)$ is decreasing in v. When $\alpha_a = -1$, $u_i(v_i', b(v_i'')) = u_i(v_i'', b(v_i'')) + v_i' - v_i''$ and $u_i(v_i'', b(v_i')) = u_i(v_i', b(v_i')) + v_i'' - v_i'$. Therefore, in equilibrium we must have $u_i(v_i', b(v_i')) = u_i(v_i'', b(v_i'')) + (v_i' - v_i'')$ but the variations of the bidding strategies are unclear.

Step 2: we now characterize the bidding strategy and show existence. When $1+\alpha_a > 0$, i's utility is:

$$u(v_i, b_i) = \int_{\underline{v}}^{b^{-1}(b_i)} (v_i - b(s)) f(s) \mathrm{d}s - \int_{b^{-1}(b_i)}^{\overline{v}} \alpha_i(v_i, s) f(s) \mathrm{d}s.$$

The optimal bid is such that

$$\frac{\partial}{\partial b_i} u(v_i, b_i) = \left[v_i - b(b^{-1}(b_i)) + \alpha_i(v_i, b^{-1}(b_i)) \right] f(b^{-1}(b_i)) b^{-1'}(b_1) = 0.$$

At the symmetric Nash equilibrium, we must have $b^{-1}(b_i) = v_i$, and therefore the optimal bidding strategy is $b(v_i) = b_i = v_i + \alpha_i(v_i, v_i)$. It exists if it is increasing, i.e. if $1 + \alpha_a + \alpha_b \geqslant 0$.

When $1 + \alpha_a < 0$, i's utility is:

$$u(v_i, b_i) = \int_{b^{-1}(b_i)}^{\overline{v}} (v_i - b(s)) f(s) \mathrm{d}s - \int_{\underline{v}}^{b^{-1}(b_i)} \alpha_i(v_i, s) f(s) \mathrm{d}s.$$

The optimal bid is such that

$$\frac{\partial}{\partial b_i} u(v_i, b_i) = -\left[v_i - b(b^{-1}(b_i)) + \alpha_i(v_i, b^{-1}(b_i)) \right] f(b^{-1}(b_i)) b^{-1'}(b_1) = 0.$$

Again, at the symmetric Nash equilibrium, we must have $b^{-1}(b_i) = v_i$, and therefore the optimal bidding strategy is $b(v_i) = b_i = v_i + \alpha_i(v_i, v_i)$.[33] It exists if it is decreasing, i.e. if $1 + \alpha_a + \alpha_b \leq 0$.

When $\alpha_a = -1$, the variations of the bidding function are unclear. If i expects his rival to use an increasing strategy, then his optimal bid is $b(v_i) = \alpha_b v_i + \gamma$. It is increasing when $\alpha_b > 0$. If i expects his rival to use a decreasing strategy, his optimal bid is again $b(v_i) = \alpha_b v_i + \gamma$ and it is decreasing when $\alpha_b < 0$. If $\alpha_b = 0$, a symmetric equilibrium is $b(v_i) = \gamma$ for all v_i. Indeed, if i expects j to bid $b(v_j) = \zeta$ for all v_j, then it is optimal to bid $b_i > \zeta$ if $\zeta < \gamma$, $b_i < \zeta$ if $\zeta > \gamma$ and any bid gives the same expected payoff when $\zeta = \gamma$. □

33 For completion, note that at equilibrium $\frac{\partial^2 u}{\partial b_i^2} \leq 0$ in both cases.

Appendix 6

Step 1: we first show that the optimal mechanism can be implemented by a modified second price sealed bid auction, only if (i) bidding strategies are pure and monotonically increasing or decreasing in valuations and (ii) the reserve price faced by each agent is monotonically increasing or decreasing in the bid of the rival.

In A^{**}, i gets the good when $v_i > r_i^{**}(v_j)$ and $v_i < v_j$. We have $v_i < v_j \Leftrightarrow b^{-1}(b_i) < b^{-1}(b_j)$. Then, $v_i < v_j \Leftrightarrow b_i > b_j$ if and only if $b(\cdot)$ is strictly decreasing at any v_i such that there is a positive probability of allocating the good to i in the optimal mechanism. In that case $v_i > r_i^{**}(v_j) \Leftrightarrow b^{-1}(b_i) > r_i^{**}(b^{-1}(b_j)) \Leftrightarrow b_i < b \circ r_i^{**} \circ b^{-1}(b_j)$. Thus the seller must allocate the good to i if $b_i < r_i(b_j)$ with $r_i = b \circ r_i^{**} \circ b^{-1}(b_j)$ and $r_i(b_j)$ is decreasing in b_j. By construction, $r_i^{-1}(\breve{b}) = r_j(\breve{b})$ at $\breve{b} = b(\breve{v})$.

Similarly, in A^*, i gets the good when $v_i > r_i^*(v_j)$ and $v_i > v_j$. In that case $b(\cdot)$ must be strictly increasing and i gets the good when $b_i > r_i(b_j)$ with $r_i = b \circ r_i^* \circ b^{-1}(b_j)$ increasing in b_j. Again, $r_i^{-1}(\breve{b}) = r_j(\breve{b})$ at $\breve{b} = b(\breve{v})$.

Step 2: We show that the seller must resort to additional entry fees. In a second price sealed bid auction, the expected payoff of agent i is

$$u_i(v_i, b(v_i)) = E_{\{v_j|i \text{ wins}\}}(v_i - b(v_j)) - E_{\{v_j|j \text{ wins}\}}\alpha_i(v_i, v_j).$$

At equilibrium, $\frac{du_i(v_i, b(v_i))}{dv_i} = \text{Prob}(i\ wins) - \alpha_a \text{Prob}(j \text{ wins})$. If the reserve prices and bidding strategies implement the optimal solution, then $\text{Prob}(i\ wins) = E_{v_j}\overline{X}_i(v_i, v_j)$ and $\text{Prob}(j\ wins)) = E_{v_j}\overline{X}_j(v_i, v_j)$ with $\overline{X}_k = \{X_k^*, X_k^{**}\}\ k = i, j$. Therefore, the expected utility in the auction is:

$$u_i(v_i, b(v_i)) = \int_{\underline{v}}^{v_i} \Big(E_{v_j}\overline{X}_i(s, v_j) - \alpha_a E_{v_j}\overline{X}_j(s, v_j)\Big)\mathrm{d}s + u_i(\underline{v}, b(\underline{v})).$$

For the payments to coincide, the seller sets c such that $u_i(v_i, b(v_i)) + c = u_i(v_i)$ where $u_i(v_i)$ is the equilibrium utility in the optimal auction (by inspecting the equations, it is easy to see that c is an amount independent of v_i). □

Appendix B

In what follows, we assume that agent i faces the reserve price $r_i(b_j)$. Also, let $\breve{b}$ be the point such that $r_i(\breve{b}) = r_j^{-1}(\breve{b})$.

Consider the case $\alpha_a \leq -1$. The reserve price is decreasing in the bid. Suppose bidder i expects the rival to bid according to a decreasing bidding function. For all $b_i < \breve{b}$, the expected payoff of bidder i is

$$u_i(v_i, b_i) = \int\limits_{b^{-1}(b_i)}^{1} (v_i - b(v_j))\mathrm{d}v_j - (\alpha_a v_i + \gamma)\Big(b^{-1}(b_i) - b^{-1}(r_j(b_i))\Big).$$

Taking the first-order condition and using the fact that $r_j(b_i) = b \circ r_j^{**} \circ b^{-1}(b_i)$, the optimal bid solves

$$v_i - b_i + \alpha_a v_i + \gamma - (\alpha_a v_i + \gamma)\frac{\mathrm{d}r_j^{**}}{\mathrm{d}v_i}_{|b^{-1}(b_i)} = 0.$$

Given $r_j^{**}(v_i) = \alpha_a v_i + \gamma$, the optimal bid conditional on bidding below $\breve{b}$ is $\underline{b}(v_i) = v_i + \alpha_a v_i + \gamma - (\alpha_a v_i + \gamma)\alpha_a$. It is decreasing in v_i and is defined only for valuations such that $v_i > \underline{b}^{-1}(\breve{b})$. For all $b_i > \breve{b}$, the expected payoff of bidder i is

$$u_i(v_i, b_i) = \int\limits_{b^{-1}(r_i^{-1}(b_i))}^{1} (v_i - b(v_j))\mathrm{d}v_j.$$

using the same technique as before, the optimal bid is $\overline{b}(v_i) = r_i^{**}(v_i)$. It is decreasing in v_i and is defined only for valuations such that $v_i < \overline{b}^{-1}(\breve{b})$. Suppose $\overline{b}^{-1}(\breve{b}) < \underline{b}^{-1}(\breve{b})$. For all $v_i < \overline{b}^{-1}(\breve{b})$,

$$\frac{\mathrm{d}u_i(v_i, \overline{b}(v_i))}{\mathrm{d}v_i} = 1 - b^{-1}(r_i^{-1}(\overline{b}(v_i))) < \frac{\mathrm{d}u_i(v_i, \breve{b})}{\mathrm{d}v_i} = 1 - b^{-1}(r^{-1}(\breve{b}))$$

therefore $u_i(v_i, \overline{b}(v_i)) \geqslant u_i(v_i, \breve{b})$ and it is optimal to bid $\overline{b}(v_i)$. For all $v_i > \underline{b}^{-1}(\breve{b})$,

$$\frac{du_i(v_i, \underline{b}(v_i))}{\mathrm{d}v_i} = (1 - v_i - \alpha_a(v_i - b^{-1}(r_j(\underline{b}(v_i))) > \frac{\mathrm{d}u_i(v_i, \breve{b})}{\mathrm{d}v_i} = 1 - b^{-1}(r^{-1}(\breve{b}))$$

therefore $u_i(v_i, \underline{b}(v_i)) \geqslant u_i(v_i, \breve{b})$ and it is optimal to bid $\underline{b}(v_i)$. A similar argument applies when $\overline{b}^{-1}(\breve{b}) > \underline{b}^{-1}(\breve{b})$. Overall, the optimal bidding strategy takes the form

$$b(v_i) = \begin{cases} \overline{b}(v_i) & v_i < \min\{\overline{b}^{-1}(\breve{b}), \underline{b}^{-1}(\breve{b})\} \\ \breve{b} & v_i \in (\min\{\overline{b}^{-1}(\breve{b}), \underline{b}^{-1}(\breve{b})\}, \max\{\overline{b}^{-1}(\breve{b}), \underline{b}^{-1}(\breve{b})\}) = M \\ \underline{b}(v_i) & v_i > \max\{\overline{b}^{-1}(\breve{b}), \underline{b}^{-1}(\breve{b})\}. \end{cases}$$

Given it is only weakly decreasing, it cannot implement the optimal mechanism. Assume the seller imposes a tax τ on bidders who obtain the good and bid below $\breve{b}$, the expected payoff of bidder i is

$$u_i(v_i, b_i) = \int_{b^{-1}(b_i)}^{1} (v_i - \tau - b(v_j))\mathrm{d}v_j - (\alpha_a v_i + \gamma)\Big(b^{-1}(b_i) - b^{-1}(r_j(b_i))\Big)$$

yielding a new bidding function $\breve{\underline{b}}(v_i) = v_i + \alpha_a v_i + \gamma - (\alpha_a v_i + \gamma)\alpha_a - \tau = \underline{b}(v_i) - \tau$, decreasing in v_i and is defined for valuations such that $v_i > \breve{\underline{b}}^{-1}(\breve{b})$. To implement the optimal mechanism, we need to choose τ such that $\breve{\underline{b}}^{-1}(\breve{b}) = \overline{b}^{-1}(\breve{b}) = \breve{v}$. In other words, $\breve{v}$ is the point at which the two bidding strategies cross. The optimal tax is $\tau = \underline{b}(\breve{v}) - \overline{b}(\breve{v})$. Overall, the optimal bidding strategy is

$$b^{**}(v_i) = \begin{cases} \overline{b}(v_i) & v_i < \breve{v} \\ \breve{\underline{b}}(v_i) & v_i > \breve{v}. \end{cases}$$

When $\alpha_a > 0$, we use the same techniques. Some of the steps are omitted. When $b_i < \breve{b}$, the optimal bidding strategy solves

$$(v_i - r_i^{-1}(b_i))\frac{\mathrm{d}r_i^{*^{-1}}}{\mathrm{d}v_i}_{|b^{-1}(b_i)} + (\alpha_a v_i + \gamma)\frac{\mathrm{d}r_j^*}{\mathrm{d}v_i}_{|b^{-1}(b_i)} = 0,$$

where $\frac{\mathrm{d}r_i^{*^{-1}}}{\mathrm{d}v_i}_{|b^{-1}(b_i)} = 1/\alpha_a$ and $\frac{\mathrm{d}r_j^*}{\mathrm{d}v_i}_{|b^{-1}(b_i)} = \alpha_a$. Differentiating the first-order condition and using the fact that r_i, r_i^* and r_j^* are increasing, the bidding strategy is also increasing. Let us denote it by $\underline{b}(v_i)$. When $b_i > \breve{b}$, the optimal bidding strategy is $\overline{b}(v_i) = v_i + \alpha_a v_i + \gamma$. It is increasing in v_i.

$$b(v_i) = \begin{cases} \underline{b}(v_i) & v_i < \min\{\overline{b}^{-1}(\breve{b}), \underline{b}^{-1}(\breve{b})\} \\ \breve{b} & v_i \in M \\ \overline{b}(v_i) & v_i > \max\{\overline{b}^{-1}(\breve{b}), \underline{b}^{-1}(\breve{b})\}. \end{cases}$$

Here again, the seller must distort the allocation to make sure the overall bidding function is strictly increasing. A possibility is to impose a tax τ on agents who obtain the good and bid above $\breve{b}$. This yields a new bidding function $\breve{\overline{b}}(v_i) = v_i + \alpha_a v_i + \gamma - \tau$. The equilibrium bidding function is

$$b^*(v_i) = \begin{cases} \underline{b}(v_i) & v_i < \breve{v} \\ \breve{\overline{b}}(v_i) & v_i > \breve{v}. \end{cases}$$

□

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