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Selling an asset to a competitor

Isabelle Brocas [a,b,*]

[a] Department of Economics, University of Southern California, 3620 S. Vermont Avenue, Los Angeles, CA 90089-0253, USA
[b] CEPR, UK

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ABSTRACT

A seller decides whether to allocate an item among two potential buyers. The seller and buyer 1 interact ex post in such a way that each of them suffers a negative externality if the other possesses the item. We show that the optimal allocation rule favors buyer 2, who does not interact ex post with the seller, and in particular bidder 1 may not obtain the good even if his valuation is highest. The auction is therefore subject to resale. When resale is possible, the seller must distort the original auction. We show that the mechanism depends crucially on the way resale is organized ex post. The seller may decide to always allocate the good to the agent with the highest valuation when rents are fully extracted by an intermediary on the resale market. However, she may resort to a stochastic mechanism when the winner of the primary auction has full bargaining power in the resale stage.

1. Motivation

Consider a firm engaged in several profitable activities. Some of them are close substitutes and are competing inefficiently against each other. The board of managers contemplates the possibility of selling a subdivision of the firm that runs one particular activity. There are two potential buyers: a direct competitor and a company that operates in a foreign market. One manager argues that selling to a competitor may be detrimental for the profitability of the remaining activities, although it is difficult to estimate the loss with accuracy. Another manager points out that a competitor may have higher stakes in avoiding competition, and may therefore be willing to pay a higher price. However, it seems that the competitor cannot assess those stakes with certainty either. The competitor would certainly pay a high price if it anticipates it will be driven out of the market in the next few years. Someone explains that behaving as if there is no hurry to sell may prompt this belief. Someone else replies that only a naive competitor would be tricked by that strategy. When they almost agree that the foreign firm would be a better choice, someone emphasizes that this will not prevent the competitor from acquiring the division: the foreign firm may sell the division in the future. Therefore, the firm may as well sell directly to its competitor or, better, keep the division.

The example above illustrates a situation common to many applications where a seller (she) decides whether to allocate an indivisible asset among several buyers (he) with whom she may interact ex post. To cite a few other examples, firms need sometimes to sell part of their assets (e.g. capital, equipment, brands, etc.) to regain financial health or simply to reorganize their activities. Assets can be transferred to competitors, or to buyers from other markets. Patent transfers or

* Corresponding author at: Department of Economics, University of Southern California, 3620 S. Vermont Avenue, Los Angeles, CA 90089-0253, USA.
E-mail address: brocas@usc.edu

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exclusive licensing agreements is another example with those features. A technology may have applications in the market in which the patentee participates and possibly in a secondary market in which it does not. Sports is another application. European soccer teams and American MLB or NFL teams may be reluctant to transfer players to other clubs competing in their same domestic championship or division. In all these cases, the owner of the asset faces a dilemma: should it sell its asset to nobody, only to firms in markets where it does not participate or can it be optimal to sell to its own competitors? The seller is likely to take the *identity* of the buyer into account to make a decision. Also, the seller faces informational asymmetries and has to make a decision based on her belief about the ability of the competitors to use the good. In particular, the extra payoff a potential acquirer may enjoy by obtaining the good is unknown to the seller, and she may also be unable to anticipate the effects of selling the good on her own payoff. This raises an interesting theoretical question: what is the optimal allocation mechanism of the item in that situation?

The examples also point to two additional issues. First, the seller's value for the asset is likely to be private information. Then, buyers face information asymmetries as well, and will make inferences from the design of the trade offer itself. The seller should account for those inferences and design an allocation mechanism accordingly. This raises the following question: how should the mechanism be designed to signal information? Second, trades between two parties are not sealed forever. The decision to allocate the item to one party can be reversed by ex post resale. Given the presence of externalities, the seller may be affected if this occurs. Then, should the seller take preventive measures to allocate the good in the first place?

The objective of the paper is to characterize the optimal allocation mechanism in these three situations. To do so, we propose the following basic model. There are three players. The first player (or seller) owns an asset that is relevant to all three players. The first and the second player are direct competitors, while the third player operates on a different market.

We first investigate the benchmark case. This corresponds to the setting in which *only buyers possess private information and there is no possibility of resale*. More precisely, the seller does not observe the willingness to pay of the bidders. Also, she does not know the level of the externality she will suffer if she decides to sell to her competitor. Given the ability to turn the asset into profit and to inflict externalities on the seller are generally linked, we assume that the intrinsic value for the good is correlated with the externality. We show that the optimal mechanism has two main elements (Proposition 1). First, the allocation rule is asymmetric and favors the bidder who does not ex post compete with the auctioneer. There are two asymmetries: then deciding whether to keep the good or sell to one of the two agents, the seller is inclined to keep the good more often when the alternative is to sell to her competitor. Then, agents face different reserve prices. When deciding whether to give the good to one of the two agents, she prefers to favor the non-competitor who is not exerting any externality on her. Then, she sometimes allocates the good to that agent even though his willingness to pay for the good is lower. Second, the presence of informational asymmetries lead the seller to increase the probability of keeping the good compared to the scenario with full information. This result is standard and reflects the usual trade-off between rent and efficiency. Note that allocation asymmetries result from the presence of asymmetric ex post interactions between the seller and the bidders. Given the seller feels differently about allocating the good to the two bidders, she will require different prices. We show in Appendix B that the mechanism can be implemented with a suitably modified second-price sealed bid auction with entry fees, ex post subsidies and different reserve prices for the different bidders.

With this in mind, we analyze the case in which *the seller is also privately informed*. Precisely, her valuation or willingness to keep the good is not observed by the buyers. Besides, her direct competitor does not know the level of the externality he will suffer if she decides to keep the good (again, because of the correlation between valuation and externality). We consider "transparent" mechanisms,[1] that is, mechanisms in which the seller offers a game form but does not participate in the subsequent message game. We characterize the general properties of the equilibrium, and we show that, at a separating equilibrium comparable to the benchmark case, the qualitative properties of the optimal mechanism described before are preserved (Propositions 2 and 4).[2] Still, the inability to observe the type of the seller affects the probability that the item changes hands differently depending on the type of goods. A direct competitor is always willing to increase his payment to induce the seller to sell when he anticipates his loss will be high otherwise. When the willingness to pay and the externality an agent inflicts on his/her competitor are positively correlated (e.g. the transfer of a drastic innovation), the seller keeps the good more often than in the benchmark case. This occurs because making trade difficult (e.g. by increasing the reserve prices) is a way to signal the externality will be high if the seller keeps the good. The double asymmetric information problem results in a further reduction in the level of trade compared to the full information case. By contrast, when the willingness to pay and the externality an agent inflicts on his/her competitor are negatively correlated (e.g. the transfer of an innovation that allows firms to differentiate their products), the seller sells the good more often when her valuation is unknown. Here, facilitating trade (by lowering the reserve prices) helps to signal that the externality will be high if the seller keeps the good. Then, the solution with double asymmetric information is less inefficient than the solution of the benchmark case.

In the last part of the paper, we extend the benchmark case to the situation where *buyers can trade ex post*. Note that resale emerges naturally because the optimal (static) auction treats bidders asymmetrically. Then, the optimal allocation

[1] This terminology was introduced in Zheng (2002). Such mechanisms are to be contrasted with mechanisms analyzed in Maskin and Tirole (1990). This will be discussed later in the analysis.

[2] Other separating and non-separating equilibria may exist.

from the perspective of the seller is sometimes not efficient from the perspective of bidders and ex post trade is beneficial. We are particularly interested in the way resale is organized. To better isolate this feature, we assume that agents' valuations are publicly disclosed after the auction, and therefore information is complete ex post, and we study two forms of resale. Either resale is organized by a third party who extracts all rents generated from trade or, the winner of the auction has full bargaining power in the resale market. We show that the possibility of resale induces the seller to distort the allocation and the way bidders organize ex post trade affects crucially her incentives. The seller can use two tools in the original mechanism. First, she can allocate the good in such a way that resale is discouraged or not. Second, she can attempt to capture part of the rents that will be generated ex post. If the rents generated in the resale stage do not accrue to bidders, then she can use only the first tool. Given the only reason to bias against the direct competitor was to avoid the externality in the benchmark case, resale makes this motive vanish. Overall, the seller sells the good to the highest valuation agent and bidders never trade ex post (Proposition 5). By contrast, if the winner can obtain rents through ex post trade, then both tools are available. There is now a motive for allocating the good to the non-competitor when his valuation is the lowest, and charge part of the extra benefit he will obtain ex post. In that case, the seller may decide to sell the good to the lower valuation agent, and resale sometimes takes place at equilibrium (Proposition 6). Or alternatively, she can replicate that outcome via a stochastic resale-proof mechanism in which the ex post transfers are adjusted to account for ex post trade.

Our study has the four following features: externalities, asymmetries, signaling and resale. It is related to four strands of the auction literature. First, the literature on auctions with externalities focuses on situations where externalities emerge ex post between bidders.[3] This corresponds to the case in which the second and third players in our game are also competitors ex post.[4] Compared to this situation, the problem when the externality is between the seller and some buyers becomes different in two dimensions: some bidders fear that the good may not be sold (rather than fearing that the good be sold to a competitor) and the auctioneer is more reluctant to allocate it to some bidders than to others. Not being able to anticipate how much can be extracted as revenue, and by how much her future payoffs will be decreased if she sells generates a new difficulty to design the mechanism. Some previous analyses suggest that externalities may exist between the seller and the bidders, but when they are modeled, they are common knowledge[5] or their presence is not the main focus of the study.[6] This paper takes a close look at this issue and therefore complements previous studies.

Second, our analysis is related to the literature on asymmetric auctions (Vickrey, 1961; Griesmer et al., 1967; Maskin and Riley, 2000). In that literature, asymmetries are due to different distributions of valuations. Moreover, the main focus is to compare how different auction formats perform. In this paper, asymmetries emerge at equilibrium and result from asymmetric ex post interactions between the seller and the bidders. Importantly, bidders do not have any interaction ex post and are ex ante symmetric vis-à-vis each other. Interestingly, similar qualitative results can emerge in those different settings. Also, we are interested in characterizing the optimal mechanism. The allocation mechanism is therefore not constrained by the existing rules and the seller is free to exploit asymmetries the way she sees fit.

Third, the case in which the seller is privately informed is related to the recent literature on signaling in auctions. Those studies characterize the optimal allocation procedure within an exogenously restricted set of mechanisms. For instance Jullien and Mariotti (2006) and Cai et al. (2007) analyze signaling problems for second-price sealed bid auctions, and characterize the optimum reserve price. In this paper, we consider transparent mechanisms, which are more general procedures as the auction designer offers a game form to select outcomes conditional on messages sent in a message sub-game.[7] The cost is that we can only provide general properties of the mechanism.

Last, our analysis is related to the literature on auctions with resale. Any allocation mechanism that treats agents asymmetrically and in such a way that the bias goes against the bidders most likely to submit high bids, is subject to resale. The seller might end up selling the item to the bidder who values it the least. Studies in the literature look at situations in which simple auction procedures lead to an inefficient outcome from the joint perspective of the bidders. Conditional on focusing on such procedures, studies determine whether efficiency can be achieved when resale is an option (see for example Gupta and Lebrun, 1999; Hafalir and Krishna, 2008; Cheng and Tan, 2007). It is shown in particular that the nature of information revelation is crucial.[8] In our environment, the designer is not bound to using a

[3] For the standard case without externalities, see Myerson (1981) for the seminal paper on optimal auctions and McAfee and McMillan (1987) and Klemperer (1999) for surveys.

[4] Several papers analyze the optimal allocation mechanism when externalities are present between bidders. See Jehiel et al. (1996) and Aseff and Chade (2008) for the case of identity-dependent externalities, Carrillo (1998) and Brocas (2008, forthcoming) for the case of type-dependent externalities.

[5] Jehiel et al. (1996) mention that it is possible to extend their analysis to the case with externalities between the bidders and the seller. However, in their setting externalities are identity-dependent but fixed and known. Lu (2006) studies a model where there are positive or negative externalities between all players. However, they are also identity-dependent and publicly disclosed ex ante.

[6] Lu (2006) is probably the only other analysis directly interested in the presence of externalities between the seller and the bidders. The paper studies the optimal auction when the seller can decide to destroy the item for sale at a cost. The focus is on when such tool will be used conditional on the configuration (and signs) of the externalities between players. It is shown in particular that it acts as a threat to induce participation when externalities are positive. Or, it can be used to collect extra payments when externalities are negative.

[7] Other studies consider the set of all possible contracts. The main illustrations are Myerson (1983) and Maskin and Tirole (1990, 1992). Following this approach, the mechanism designer also participates in the message sub-game and the mechanism is contingent on the agents' and her own reports.

[8] In Gupta and Lebrun (1999), the values are publicly disclosed after the first-price auction. Then bidders know their respective valuations and resale leads to efficiency. In Hafalir and Krishna (2008), only bids are disclosed, then it is optimal to bid in such a way that the true value is not revealed. Also, resale might not lead to efficiency. See also Haile (2003) and Garratt and Tröger (2006) for analyses of auctions with resale in alternative settings.

particular format and we are interested in determining the optimal mechanism under the additional resale constraint. In that respect, our work is closer in spirit to Zheng (2002) and Calzolari and Pavan (2006). Both works consider an environment à la Myerson (valuations are drawn from different distributions) and study the design of primary auctions when information is not exogenously revealed before the secondary market opens. In that setting, the resale outcome can be influenced by resorting to a disclosure policy that optimally shapes the beliefs of the players. Zheng (2002) characterizes conditions under which the initial seller can still achieve the optimal auction à la Myerson when the owner of the good has full bargaining power. Calzolari and Pavan (2006) show that this fails when the owner of the good does not have full bargaining power. The present paper studies a different setting. In particular, it departs from those analyses by assuming that information is complete ex post and focuses on the effect of the externalities between parties. In particular, given the presence of externalities, the seller has preferences over the allocation. To our knowledge, this is the first study of resale when the seller in the primary market is an interested party. Our study also offers a different but complementary perspective. Indeed, we are not interested in optimal disclosure policies (given information is disclosed, the party with the highest valuation will always get the item) but we emphasize the effect of the resale format on the design of the primary auction and on the ability to secure revenue.

2. The model

2.1. A simple model with asymmetric ex post interactions

We consider a stylized model that captures a few features of interest. Some variants are discussed later in the text (see Remarks 1 and 2). There are three risk neutral agents $i=\{0,1,2\}$ who derive known positive payoffs normalized to $\overline{\phi}$. Agent $i=0$ ('she') possesses an indivisible good that she intends to sell. She is the seller. Agents $i=\{1,2\}$ (each referred to as 'he') are two potential buyers who compete for the obtention of the good. The item is an asset that will be used subsequently. Possessing the item translates into an ability or efficiency level θ_i, also called type. This ability is used by the owner of the item to increase his payoff. The interactions between agents are asymmetric: the seller and buyer 1 compete on the same market, while buyer 2 operates alone on a different market. Therefore, there are three possible ex post outcomes. If the seller keeps the good, ex post payoffs are affected by θ_0 and can be summarized by the functions $\phi_0^0(\theta_0)>\overline{\phi}$ and $\phi_1^0(\theta_0)<\overline{\phi}$ for the seller and buyer 1, respectively. If the seller allocates the good to buyer 1, ex post payoffs depend on θ_1 and are summarized by $\phi_0^1(\theta_1)<\overline{\phi}$ and $\phi_1^1(\theta_1)>\overline{\phi}$ for the seller and buyer 1, respectively. If the seller allocates the good to buyer 2, his ex post payoff is $\phi(\theta_2)>\overline{\phi}$.

2.2. Values and externalities

Each agent's value of the good depends on whether he/she possesses it ex post and, on who possesses it if he/she does not. There are two types of values. First, each agent i gets a variation in payoff v_i when he/she possesses the good. We call it his "valuation". Formally, $v_0\equiv\phi_0^0(\theta_0)-\overline{\phi}$, $v_1\equiv\phi_1^1(\theta_1)-\overline{\phi}$ and $v_2=\phi(\theta_2)-\overline{\phi}$. It is profitable to obtain the good and $v_i>0$ for all i. Second, some agents get a variation in payoff when a competitor possesses the good ex post. The quantity $\alpha_i(v_j)\equiv\beta_i(\theta_j)\equiv\overline{\phi}-\phi_i^j(\theta_j)$ represents the "externality" on agent $i\in\{0,1\}$ when agent $j\in\{0,1\}$ obtains the item. Note that $\alpha_j(v_i)>0$ for all $i\in\{0,1\}$, that is externalities are negative. Bidder 2 neither exerts nor suffers an externality, there are externalities between the seller and bidder 1 only and there is no externality between bidders.

We assume that $\phi_0^1(\cdot)=\phi_1^0(\cdot)$ and $\phi_0^0(\cdot)=\phi_1^1(\cdot)=\phi(\cdot)$. Also, types θ_1 and θ_2 are unknown and drawn independently from the same distribution. Therefore, valuations v_1 and v_2 are independent and take values on a support $[\underline{v},\overline{v}]$ with $\underline{v}>0$. The c.d.f. is denoted by $F(v_i)$ and the positive density by $f(v_i)$. Moreover, externalities are symmetric ($\alpha_0(\cdot)=\alpha_1(\cdot)=\alpha(\cdot)$), which allows us to concentrate on qualitative properties of our study (see Remark 2 for a discussion) and C^1 on $[\underline{v},\overline{v}]$. Last, the type of the seller may or may not be known. In Sections 3 and 5, the type θ_0 of the seller is common knowledge, hence v_0 is known. In Section 4, bidders do not observe θ_0 or v_0[9] and the prior beliefs of bidders over the seller's valuation are summarized by the cumulative probability distribution $G(\cdot)$ with density $g(\cdot)$ on the support $[\underline{v},\overline{v}]$. Note that given the presence of the externality between the seller and agent 1, the outside option of agent 1 depends on the mechanism that is specified when he does not show up. We normalize any other payoffs agents 1 and 2 might obtain to 0.

2.3. Examples

We concentrate on situations where (i) the good for sale is not divisible and can be owned by at most one agent, (ii) it generates a positive value for its owner and,[10] and (iii) the seller cares about the identity of the agent who ends up possessing the item. Below are a few examples.

[9] In that case, there is double asymmetric information and, given the seller is informed, the mechanism might reveal some relevant information to bidders. They may decide in turn to act upon it.

[10] This means in particular that we do not consider situations where the owner has to sell because he cannot use the good anymore, or the good itself is obsolete to its owner.

Selling a location. Major superstores (e.g. Target, WalMart) sometimes decide to move their locations. When this happens, customers need to find an alternative. Suppose that a store (agent 0) wants to relocate and abandon a location in a mall. If it sells the location to a store that offers similar products (agent 1), its clients may purchase from this new store. If, on the contrary, it sells the location to a store offering different products (agent 2), its clients may drive to the new location. It may also be the case that, if the location is retained, agent 1 cannot open a profitable store in that mall (e.g. because a second large warehouse cannot be accommodated) and an increasing number of customers will buy from agent 0.[11]

Selling a sports player. Sports teams often need to sell their players. Each time a player is on the market, there are generally various alternatives ranging from selling the player to a team that competes in the same domestic tournaments or divisions (agent 1), to selling him to a foreign team or a team in a different division (agent 2). As externalities are involved in the first alternative, teams may be reluctant to transfer players to other clubs competing in their same domestic championship or division. As an anecdote, in the summer of 2001, two soccer teams Real Madrid C.F. (from Spain) and S.S. Lazio (from Italy) were bidding for Gaizka Mendieta, a midfielder of Valencia C.F. (also from Spain). Interestingly, according to the sports press, Valencia turned down a EUR 46 M offer of Real Madrid to accept a EUR 36 M offer of Lazio. Moreover, the contract specified that the buyer would have to pay an extra EUR 12 M if the player were transferred within two years to another Spanish team.

Transferring a patent. Innovation transfers are well known to generate externalities between firms. In our case, suppose for instance that a firm obtained a patent for an innovation. It can sell and transfer the patent to a major competitor (agent 1), or to a small firm that operates on a different market (agent 2). If the patent is not transferred, the firm will use it and offer a better product than agent 1, which will be hurt eventually.

We want to understand the motivations of the seller when she allocates the good. We analyze our benchmark case in Section 3. Yet, depending on the situation, the seller may need to take specific issues into considerations. We will focus on two of them. First, selling an asset may convey information that is not accessible otherwise. This means that the value of the item to the seller is not necessarily observable to potential buyers.[12] We analyze this case in Section 4. Second, there may be a secondary market where potential buyers trade ex post. Given the seller cares about the identity of the agent who ends up owning the good, she cares about potential detrimental reallocations.[13] We study such situations in Section 5.

3. Allocation mechanisms in the benchmark case

In the benchmark case, only valuations v_1 and v_2 are private information. We apply the general procedure introduced by Myerson (1981). The auction mechanism consists of a message space for each buyer, winning probabilities and transfers to the seller. Applying the revelation principle for Bayesian games,[14] we can restrict the attention to a Bayesian equilibrium for a direct mechanism that induces truth-telling.[15] A direct mechanism $\mathcal{A}$ is characterized by the interim probability $X_i(v)$ that bidder i gets the good and the transfer $t_i(v)$ from bidder i to the seller, which are both function of the vectors of valuations of both bidders $v \equiv (v_1,v_2)$. Also, $X_0(v)=1-X_1(v)-X_2(v)$ is the interim probability that the seller keeps the good.

Let $u_i(v_i,\tilde{v}_i)$ be the expected *utility* of bidder i when he participates in the auction, his valuation is v_i, he announces $\tilde{v}_i$, and the other bidder discloses his true valuation. We denote by $u_i(v_i) \equiv u_i(v_i,v_i)$ his expected utility under truthful revelation. We have

$$u_1(v_1,\tilde{v}_1)=E_{v_2}[v_1X_1(\tilde{v}_1,v_2)-\alpha(v_0)X_0(\tilde{v}_1,v_2)-t_1(\tilde{v}_1,v_2)], \tag{1}$$

$$u_2(v_2,\tilde{v}_2)=E_{v_1}[v_2X_2(v_1,\tilde{v}_2)-t_2(v_1,\tilde{v}_2)]. \tag{2}$$

To be feasible, the mechanism must satisfy three kinds of constraints. First, the mechanism must be *incentive compatible*, that is such that each bidder finds profitable to report truthfully his true valuation. Formally, we must have $u_i(v_i) \geq u_i(v_i,\tilde{v}_i)$ for all v_i, $\tilde{v}_i$ and $i \in \{1, 2\}$. Second, it must be *individually rational* to participate, that is agents must obtain a higher payoff when they show up. If we denote by w_i the *reservation utility* of bidder i, the mechanism must be such that $u_i(v_i) \geq w_i$ for all $i \in \{1, 2\}$. Note that we have by assumption $w_2=0$. However, the outside option of agent 1 depends on who is allocated the good if he decides not to show up: either agent 2 gets the good and agent 1's payoff is 0, or the seller keeps the good and agent 1's payoff is $-\alpha(v_0)$. It has already been shown in the literature of auctions with externalities between bidders that it is optimal for the seller to threaten buyer 1 with his worst outside option in case of declining participation.[16] The same logic applies here. In the optimal mechanism, the seller commits to keep the good with probability 1 if agent 1 does not participate, implying that agent 1 suffers the externality for sure in that case. Therefore $w_1=-\alpha(v_0)$. Also, exerting that

[11] Similar examples include selling a brand, a division, or equipment during the process of restructuration.

[12] This is probably most relevant in the case of the sale of a company or a patent.

[13] This is obviously true in the case of sports players, or brands, or more generally any item for which resale markets and practices are easily available.

[14] See Myerson (1979).

[15] For the sake of brevity, we skip some of the formal proofs that are standard in this literature.

[16] See for instance Jehiel et al. (1996). Brocas (2003) analyzes a specific procedure when this commitment ability is absent and shows there is a coordination problem in the decision to participate.

option generates the positive payoff v_0 for the seller. By committing to keep the good, the seller only gives up the possibility of increasing her payoff by selling to buyer 2.[17] Also, the threat is costless, since it occurs only out-of-equilibrium. Last, the allocation rule must be *feasible*, $X_0(v)\geq 0$, $X_1(v)\geq 0$, $X_2(v)\geq 0$ and $X_0(v)+X_1(v)+X_2(v)=1$ for all v. An optimal direct mechanism solves $\mathcal{P}$:

$$\mathcal{P}: \max \quad R_0=\int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{\overline{v}}[t_1(v)+t_2(v)+v_0X_0(v)-\alpha(v_1)X_1(v)]\,f(v_1)f(v_2)\,dv_1\,dv_2$$

$$\text{s.t.}\quad \begin{array}{ll} u_i(v_i)\geq u_i(v_i,\tilde{v}_i)\quad \forall i,v_i,\tilde{v}_i & (\mathrm{IC_i})\\ u_i(v_i)\geq w_i\quad \forall i,v_i & (\mathrm{IR_i})\\ X_0(v)\geq 0,\quad X_1(v)\geq 0,\quad X_2(v)\geq 0,\quad X_0(v)+X_1(v)+X_2(v)=1,\quad \forall v & (\mathrm{F}) \end{array}$$

where R_0 is the expected utility of the seller, ($\mathrm{IC_i}$) and ($\mathrm{IR_i}$) are the standard incentive-compatibility and individual-rationality constraints for each bidder $i=\{1, 2\}$ and (F) ensures the feasibility of the allocation rule.

Let $\pi_1^*(v_1)=v_1-\alpha(v_1)+\alpha(v_0)-v_0-(1-F(v_1))/f(v_1)$ and $\pi_2^*(v_2)=v_2+\alpha(v_0)-v_0-(1-F(v_2))/f(v_2)$.

Lemma 1. *The seller's optimization program can be rewritten as*

$$\mathcal{P}^*: \max \quad \int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{\overline{v}}[X_1(v)\pi_1^*(v_1)+X_2(v)\pi_2^*(v_2)]\,dF(v_1)\,dF(v_2)+v_0$$

$$\text{s.t.}\quad \begin{array}{ll} E_{v_j}X_i(v_i',v_j)\leq E_{v_j}X_i(v_i,v_j)\quad \forall v_i'\leq v_i\quad \forall\{i,j\}=\{1,2\} & (\mathrm{M})\\ X_0(v)\geq 0,\quad X_1(v)\geq 0,\quad X_2(v)\geq 0,\quad X_0(v)+X_1(v)+X_2(v)=1\quad \forall v & (\mathrm{F}) \end{array}$$

Proof. See Appendix A.

Deriving this result is standard. Using the usual terminology (Myerson, 1981), $\pi_i^*(v_i)$ is agent i's *virtual surplus*. It is the net surplus that the auctioneer can extract by selling the good to i rather than keeping it, adjusted for the informational rents.[18] By selling to agent 1, the seller can obtain what agent 1 is willing to pay: v_1 to obtain the good and $\alpha(v_0)$ to prevent the seller from keeping it. By keeping the good, the seller obtains a modified value: her intrinsic value v_0 and the option of not suffering the externality $\alpha(v_1)$. The net surplus from selling to agent 1 is therefore $v_1+\alpha(v_0)-(v_0+\alpha(v_1))$ and the virtual surplus reflects the extra informational rents. Consider now allocating the item to agent 2. The interesting feature is that agent 1 is ready to 'bribe' the seller for not keeping the good up to $\alpha(v_0)$. By selling to agent 2, the seller's net surplus is $v_2+\alpha(v_0)-v_0$. Again, the virtual surplus adjusts for informational rents. Last, (M) is the standard monotonicity condition.

Assumption 1. $\frac{d}{ds}\left[\frac{1-F(s)}{f(s)}\right]\leq \min\{0,1-\alpha'(s)\}$ for all s.

Assumption 1 guarantees that both virtual surplus are well behaved, increasing in their arguments. Note in particular that under Assumption 1, the surplus extracted from selling the good to agent 1 increases faster than the option of keeping the good. Define

$$h(v_1)=\min\{v_2\,|\,\pi_1^*(v_1)\leq \pi_2^*(v_2)\}.$$

It is increasing in v_1 and $h(v_1)<v_1$. Also there exists $v_1^{\min}(>\underline{v})$ such that $\pi_1^*(v_1)<\pi_2^*(v_2)$ for all $v_1<v_1^{\min}$ and for all v_2. Last, let $r_1^*=\overline{v}$ if $\pi_1^*(v_1)<0$ for all v_1 and $r_1^*=\max\{\operatorname{argmin}\{v_1\,|\,\pi_1^*(v_1)\geq 0\};v_1^{\min}\}$ otherwise. Also, let $r_2^*=\overline{v}$ if $\pi_2^*(v_2)<0$ for all v_2 and $r_2^*=\max\{\operatorname{argmin}\{v_2\,|\,\pi_2^*(v_2)\geq 0\};\underline{v}\}$ otherwise. For all $r_1^*\in[v_1^{\min},\overline{v})$, we have $r_2^*=h(r_1^*)$.

Proposition 1. *Under Assumption 1, the optimal mechanism $\mathcal{A}^*$ entails*:

$$\begin{array}{ll} X_1^*(v)=1 & \text{if } v_1\geq \max(r_1^*,h^{-1}(v_2)\},\\ X_2^*(v)=1 & \text{if } v_2\geq \max\{r_2^*,h(v_1)\},\\ X_0^*(v)=1 & \text{otherwise}. \end{array}$$

(i) *The agent with highest valuation does not necessarily get the good* ($h(v_1)<v_1$ *hence* $v_2\in(h(v_1),v_1)\Leftrightarrow X_1^*(v)=0$ *and* $X_2^*(v)=1$).
(ii) *The reserve price for agent1 is higher than for agent2* ($r_1^*>r_2^*$).
(iii) *The good is allocated less often than under full information.*

[17] In other settings, the commitment assumption may require to use threats that are less credible, such as giving the good for free.

[18] In particular, for each unit of rent left to type v_i (with probability $f(v_i)$), she needs to leave a unit of rent to each type above v_i (in proportion $1-F(v_i)$).

(iv) *The expected utility of bidders 1 and 2 are*

$$u_1(v_1) = \int_{\underline{v}}^{v_1} E_{v_2} X_1^*(s,v_2)\, ds - \alpha(v_0), \quad u_2(v_2) = \int_{\underline{v}}^{v_2} E_{v_1} X_2^*(s,v_1)\, ds.$$

Proof. See Appendix A.

The optimal allocation mechanism is *asymmetric*. Given the externalities, the seller is not inclined to sell to agent 1, and agent 1 is willing to pay to avoid that the seller keeps the good, even if it means it is sold to bidder 2. Therefore, agent 2 is favored by the allocation mechanism ($h(v_1) < v_1$ and point (ii)). As a consequence, agent 1 sometimes does not get the good even though his valuation is highest (point (i)). Interestingly, allocating the good to agent 2 is an extra tool for the seller to avoid the externality and a reasonable compromise for agent 1. Last, as usual in mechanism design problems under asymmetric information, the allocation is inefficient (point (iii)).[19] Our results are summarized in Fig. 1.

Remark 1. Allowing externalities to be asymmetric would modify the results only quantitatively. To be more precise, if we modify the virtual surpluses by replacing $\alpha(v_0)$ by $\alpha_1(v_0)$ and $\alpha(v_1)$ by $\alpha_0(v_1)$, we still obtain the same properties in Proposition 1. Only, the magnitude of these properties are affected. Remember that the seller responds primarily to the fact that selling to buyer 1 generates an externality $\alpha(v_1)$, whereas selling to buyer 2 does not.

Remark 2. We may think of other competitive situations yielding a different payoff structure. We have assumed that the underlying uncertainty θ_i is related to the ability of the asset owner to make use of it. Then, the externality suffered by an agent depends exclusively on the type of his rival when the latter owns the asset. In other applications, however, the externality suffered by an agent may also depend on his own type. Such situations have been discussed and studied in Brocas (forthcoming) for the case of externalities between bidders. In our setting, the externality suffered by agent *i* would become a function $\alpha_i(v_i,v_j)$. The relationship between the externalities and the seller's type would not affect our results qualitatively as long as v_0 is known. However, and as shown in Brocas (forthcoming), the incentives to reveal truthfully of bidder 1 would be modified: the agent may have incentives to hide his type not only to prevent the seller from assessing his willingness to pay for the asset v_1, but also the externality level he truly suffers when the seller keeps it $\alpha_1(v_0,v_1)$. Adding such extra features is out of the scope of the present paper, as the effects are orthogonal to the problem at hand.

4. Optimal auction when the seller's valuation is unknown

We now assume that v_0 is private information. We consider the following contracting game. At stage 1, the seller designs the auction $\mathcal{A}$. At stage 2, buyers decide whether to participate. At stage 3, each bidder sends a message. The mechanism is implemented accordingly.[20] The revelation principle for Bayesian games applies: for any mechanism and for given beliefs after the bidders have decided to participate, any equilibrium of the mechanism corresponds to an equilibrium of a direct revelation mechanism in which announcements are truthful.[21] After observing the mechanism, bidders revise their beliefs. Let $\gamma(\mathcal{A}) = E[\alpha(v_0)|\mathcal{A}]$. Let $\mathcal{A}^*(v_0) = (X_1^*(v;v_0), X_2^*(v;v_0), t_1^*(v;v_0), t_2^*(v;v_0))$ be the mechanism a seller with valuation v_0 offers in Section 3. The expected utility of bidder 1 with type v_1 and report v_1' is

$$u_1(v_1,v_1',\gamma(\mathcal{A})) = E_{v_2}[v_1 X_1(v_1',v_2) - \gamma(\mathcal{A}) X_0(v_1',v_2) - t_1(v_1',v_2)], \tag{3}$$

while his outside option is now $\tilde{w}_1 = -\gamma(\mathcal{A})$. We denote by $u_1(v_1,\gamma(\mathcal{A})) \equiv u_1(v_1,v_1,\gamma(\mathcal{A}))$ his expected utility under truthful revelation. As for bidder 2, we have

$$u_2(v_2,v_2') = E_{v_1}[v_2 X_2(v_1,v_2') - t_2(v_1,v_2')]. \tag{4}$$

We concentrate on the case with *common values*, obtained when $\alpha'(\cdot) \neq 0$: bidder 1 is affected by the information of the seller and his beliefs after observing the mechanism.[22]

4.1. General properties of the optimal mechanism with signaling

We look for a perfect Bayesian equilibrium, i.e. a strategy for each party (agents and seller) and a belief such that all strategies are optimal given the belief and the belief is consistent, given the strategies. The mechanism must be incentive

[19] Note that the negative outside option is reflected in the expression of the equilibrium rent of bidder 1. If his type is below r_1^*, he does not obtain the good and his expected payoff is $-\alpha(v_0)$. This means in particular that the seller designs a payment scheme such that bidder 1 has to pay even if he does not acquire the good.

[20] The inability of the seller to send further messages should be understood as a restriction on the set of possible contracts.

[21] See Myerson (1979).

[22] We have *private values* when $\alpha'(\cdot) = 0$: v_0 does not affect directly the utility or the outside option of bidder 1. Bidders do not use strategically the information contained in the mechanism. A seller with type v_0 offers $\mathcal{A}^*(v_0)$ and v_0 is fully deduced. Note that in Maskin and Tirole (1990), the seller is also allowed to send a message in the message game. Therefore, the agent must take expectations with respect to the principal's message. This implies that (and contrary to our case) it is necessary to satisfy incentive compatibility and individual rationality only on average. This makes the principal better-off when her type is unknown.

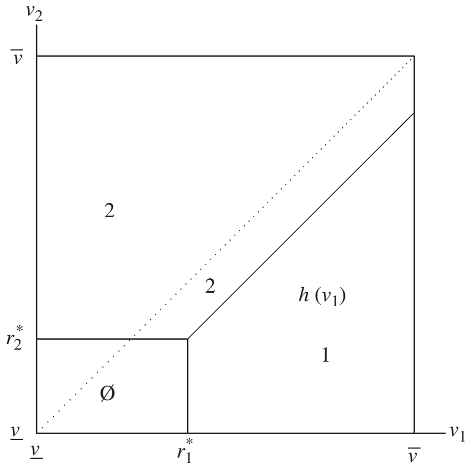

Fig. 1. Optimal allocation in the benchmark case.

compatible, individually rational, feasible and consistent with the posterior belief. We need to satisfy the following Bayesian rationality condition $\gamma(\mathcal{A}) = E[\alpha(v_0)|\mathcal{A}]$, denoted by (B). The optimization program is now:

$$\mathcal{P}^{\mathcal{I}} : \max \quad R_0^I = \int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{\overline{v}} [t_1(v)+t_2(v)+v_0\, X_0(v) - \alpha(v_1)X_1(v)] f(v_1) f(v_2)\, dv_1\, dv_2$$

$$\text{s.t.} \quad \begin{array}{ll} u_1(v_1,\gamma(\mathcal{A})) \geq u_1(v_1,\tilde{v}_1,\gamma(\mathcal{A})) \quad \forall v_1,\tilde{v}_1 & (\mathrm{IC}_1) \\ u_2(v_2) \geq u_2(v_2,\tilde{v}_2) \quad \forall v_2,\tilde{v}_2 & (\mathrm{IC}_2) \\ u_1(v_1,\gamma(\mathcal{A})) \geq -\gamma(\mathcal{A}) \quad \forall v_1 & (\mathrm{IR}_1) \\ u_2(v_2) \geq 0 \quad \forall v_2 & (\mathrm{IR}_2) \\ X_0(v) \geq 0, \quad X_1(v) \geq 0, \quad X_2(v) \geq 0, \quad X_0(v)+X_1(v)+X_2(v) = 1 \quad \forall v & (\mathrm{F}) \\ \gamma(\mathcal{A}) = E[\alpha(v_0)|\mathcal{A}] & (\mathrm{B}) \end{array}$$

Let $\tilde{\pi}_1(v_1; v_0) = v_1 - \alpha(v_1) + \gamma(\mathcal{A}) - v_0 - (1-F(v_1))/f(v_1)$ and $\tilde{\pi}_2(v_2; v_0) = v_2 + \gamma(\mathcal{A}) - v_0 - (1-F(v_2))/f(v_2)$.

Lemma 2. *The seller's optimization program is equivalent to*:

$$\mathcal{P}^{\mathcal{I}} : \ \max \quad \int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{\overline{v}} [X_1(v)\tilde{\pi}_1(v_1; v_0) + X_2(v)\tilde{\pi}_2(v_2; v_0)]\, dF(v_1)\, dF(v_2) + v_0$$

$$\text{s.t.} \quad \begin{array}{ll} E_{v_j} X_i(v_i', v_j) \leq E_{v_j} X_i(v_i, v_j) \quad \forall v_i' \leq v_i, \quad \forall \{i,j\} = \{1,2\} & (\mathrm{M}) \\ X_0(v) \geq 0,\ X_1(v) \geq 0,\ X_2(v) \geq 0,\ X_0(v)+X_1(v)+X_2(v) = 1 \quad \forall v & (\mathrm{F}) \\ \gamma(\mathcal{A}) = E[\alpha(v_0)|\mathcal{A}] & (\mathrm{B}) \end{array}$$

Proof. Neglecting (B), the proof follows the same lines as in Lemma 1.[23] □

The new virtual surplus reflect the fact that agent 1 must take expectations about the type of the seller (using the revised distribution). The solution of $\mathcal{P}^{\mathcal{I}}$ is a mechanism $\mathcal{A}(v_0) = (X_1(v; v_0), X_2(v; v_0), t_1(v; v_0), t_2(v; v_0))$. We shall rewrite the revenue of the seller as a function of her true type v_0, the expected externality γ, and the mechanism $\mathcal{A}$ she selects, $R(v_0,\gamma,\mathcal{A})$. The solution of $\mathcal{P}^{\mathcal{I}}$ satisfies the sequential rationality condition (SR):

$$R(v_0,\hat{\gamma}(\mathcal{A}(v_0)),\mathcal{A}(v_0)) \geq R(v_0,\hat{\gamma}(\mathcal{A}),\mathcal{A}) \quad \forall \mathcal{A},\ v_0, \tag{SR}$$

where the belief inferred from the observation of the mechanism is a function $\hat{\gamma}(\mathcal{A})$ that satisfies (B), i.e. $\hat{\gamma}(\mathcal{A}) = E[\alpha(v_0)|\mathcal{A}(v_0) = \mathcal{A}]$.

Proposition 2. *The optimal mechanism with signaling satisfies the following properties*:

(i) *The equilibrium revenue is non-decreasing in v_0 and for all $v_0 > v_0'$,*
$R(v_0,\gamma(\mathcal{A}(v_0)),\mathcal{A}(v_0)) - R(v_0',\gamma(\mathcal{A}(v_0')),\mathcal{A}(v_0')) = \int_{v_0'}^{v_0} \int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{\overline{v}} [X_0(v; s)]\, dF(v_1)\, dF(v_2)\, ds$.

[23] At this stage, we also assume that the optimal mechanism can be supported by off-path beliefs. We will show that such beliefs exist in the characterization of separable allocations.

(ii) *The probability that the seller keeps the good is non-decreasing in* v_0.
(iii) *The beliefs are monotone functions of* v_0.

Those conditions are not satisfied by the benchmark optimum $\mathcal{A}^*(v_0)$.

Proof. See Appendix A.

Sequential rationality implies that a seller with type v_0 picks the mechanism that maximizes her revenue. Such mechanism is such that the seller is more likely to keep the good when her type increases (point (ii)), and at equilibrium, a seller with a higher type must have a higher revenue (point (i)) as she can mimic a seller with a lower type. At equilibrium, the seller offers a mechanism from which the probability of keeping the good can be inferred and a higher probability of keeping the good is a signal that v_0 is high (point (iii)). Still, for agent 1, observing a mechanism yields good or bad news depending on the relationship between the seller's type and the externality suffered. When $\alpha'(\cdot) > 0$, inferring a high probability of keeping the good is interpreted as bad news: the type of the seller is likely to be high and the externality will be also high. The contrary holds when $\alpha'(\cdot) < 0$. In that case, inferring a high probability of keeping the good is interpreted as good news: the type of the seller is likely to be high and the externality will therefore be low. Last, as usual in signaling problems, the benchmark optimum is not sequentially rational.

4.2. Separating allocations

The properties highlighted in Proposition 2 hold in any equilibrium of our signaling game. To be able to compare the solution with the benchmark case, we concentrate thereafter on separating allocations, that is $\gamma = \alpha(v_0)$ at equilibrium.

Proposition 3. *In any separating allocation, the revenue of the seller satisfies*:

(i) *when* $\alpha'(\cdot) > 0$: $R(v_0, \alpha(v_0), \mathcal{A}(v_0)) = \int_{\underline{v}}^{v_0} \int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{\overline{v}} [X_0(v;s)]\, dF(v_1)\, dF(v_2)\, ds + R(\underline{v}, \alpha(\underline{v}), \mathcal{A}^*(\underline{v}))$,
(ii) *when* $\alpha'(\cdot) < 0$: $R(v_0, \alpha(v_0), \mathcal{A}(v_0)) = R(\overline{v}, \alpha(\overline{v}), \mathcal{A}^*(\overline{v})) - \int_{v_0}^{\overline{v}} \int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{\overline{v}} [X_0(v;s)]\, dF(v_1)\, dF(v_2)\, ds$.

Proof. See Appendix A.

When $\alpha'(\cdot) > 0$, $\gamma = \alpha(\underline{v})$ is the worst belief the uninformed agent can hold. If a principal with type $\underline{v}$ offers $\mathcal{A}(\underline{v}) \neq \mathcal{A}^*(\underline{v})$, the equilibrium belief function $\hat{\gamma}(\cdot)$ is such that the seller is perceived to be a $\underline{v}$-type, and she obtains a lower payoff compared to the benchmark. Now, if she were to offer $\mathcal{A}^*(\underline{v})$ instead, she would be perceived to induce an externality level $\hat{\gamma}(\mathcal{A}^*(\underline{v})) \geq \alpha(\underline{v})$ and she would increase her payoff. Overall, sequential rationality is consistent with the boundary condition $\mathcal{A}(\underline{v}) = \mathcal{A}^*(\underline{v})$. Similarly, when $\alpha'(\cdot) < 0$, $\gamma = \alpha(\overline{v})$ is the worst belief the uninformed agent can hold and the boundary condition is $\mathcal{A}(\overline{v}) = \mathcal{A}^*(\overline{v})$.

A general procedure to characterize separating equilibria is due to Riley (1979) and Mailath (1987). Usually, the strategy space of the informed party consists of actions.[24] In this analysis however, it consists of contracts. Despite the technical differences, the results we obtained are similar. Indeed, (SR) is consistent with a boundary condition: a seller with a type yielding the worst possible belief chooses the benchmark allocation. Also, (SR) implies that the payoff must be increasing in type. This is usually true because of an underlying single crossing condition guaranteeing the indifference curves of the informed party in the belief-action space cross only once. In our case, the problem of the seller is mostly to choose the probability of keeping the good. Reducing the contract to $E[X_0(v;v_0)]$, there is an intuitive single crossing condition: when the belief becomes more favorable, in which case the seller can extract larger payments from bidder 1 if she sells, she must decrease the probability of selling to keep the payoff constant. That is, indifference curves are monotonic. Also, the effect of an increase in belief is less important as the true valuation is high and the slope of the indifference curves decrease in v_0. Therefore, curves cross only once.

Let $r_i^S(v_0)$ be the reserve price faced by agent i in such an allocation. To compare the solution with the benchmark case, we concentrate on particular separating allocations such that $h(r_1^S(v_0)) = r_2^S(v_0)$ for all $r_1^S(v_0) \in [v_1^{\min}, \overline{v})$.[25] With this restriction, the problem of the seller is to choose one reserve price. The problem falls now into the case studied in Mailath (1987).

[24] See Jullien and Mariotti (2006) and Cai et al. (2007) for such analyses.
[25] Other separating allocations may exist, in particular some in which $h(r_1^S(v_0)) \neq r_2^S(v_0)$.

Proposition 4. *There exists a unique separating mechanism* $\mathcal{A}^S(v_0)$ *such that*

$$X_1^S(v;v_0)=1 \quad \text{if } v_1 \geq \max(r_1^S(v_0),h^{-1}(v_2)\},$$
$$X_2^S(v;v_0)=1 \quad \text{if } v_2 \geq \max\{r_2^S(v_0),h(v_1)\},$$
$$X_0^S(v;v_0)=1 \quad \text{otherwise},$$

such that $h(r_1^S(v_0))=r_2^S(v_0))$.

(i) *When* $\alpha'(\cdot)>0$, *the seller allocates the good less often compared to the case where her valuation is known* $(X_0^S(v;v_0)>X_0^*(v;v_0))$.
(ii) *When* $\alpha'(\cdot)<0$, *the seller allocates the good more often compared to the case where her valuation is known* $(X_0^S(v;v_0)<X_0^*(v;v_0))$.
(iii) *The equilibrium mechanism can be sustained with out-of-equilibrium beliefs* $\gamma(\mathcal{A})=\alpha(\underline{v})$ *when* $\alpha'>0$ *and* $\gamma(\mathcal{A})=\alpha(\overline{v})$ *when* $\alpha'<0$ *for any* $\mathcal{A}\notin(\mathcal{A}^S(v_0))_{v_0\in[\underline{v},\overline{v}]}$.

Proof. See Appendix A.

Compared to the benchmark case, there is no reason to distort the allocation rule between bidder 1 and bidder 2. The seller will only distort the reserve prices faced by the agents[26] and the distortion depends on the shape of the externality. When $\alpha'(\cdot)>0$, the benchmark revenue increases too fast in v_0 and restoring sequential rationality requires to *keep the good more often*. By contrast, when $\alpha'(\cdot)<0$, the benchmark revenue increases too slowly in v_0 and sequential rationality requires to *sell the good more often*.[27] The inability to observe the type of the seller affects the probability that the item changes hands differently depending on the type of goods. In the case of innovation transfers and if the innovation is drastic ($\alpha'(\cdot)>0$), the seller keeps the good more often when her valuation is unknown because she cannot induce bidder 1 to pay enough for her to find profitable to sell to either bidder. The double asymmetric information problem results in a further reduction in the level of trade compared to the full information case. Sellers prefer to keep their assets because information asymmetries prevent to price them correctly. By contrast, if the innovation allows firms to differentiate their products ($\alpha'(\cdot)<0$), the seller sells the good more often when her valuation is unknown. Lowering the price is a way to signal its low intrinsic value and inform bidder 1 the externality will be high if the seller keeps the good. The double asymmetric information problem now results in an increased level of trade compared to the benchmark case and a reduction of the inefficiencies.[28] Last, the mechanism can be sustained with pessimistic beliefs, as usual in signaling models.

5. Allocation mechanism with resale

In Section 3, we have demonstrated that an asymmetric allocation emerges at equilibrium. This asymmetry gives rise naturally to a resale problem. A central question is whether the seller still wants to offer an asymmetric rule. To address this issue, we assume that bidders can meet ex post and trade. We also suppose that valuations are publicly revealed ex post, at least to bidders 1 and 2. Even though this assumption is restrictive, it allows us to concentrate on the problem of resale and to abstract from any information leakage in the auction.[29] As will become clear in the next paragraphs, we are interested in comparing different resale market environments, and their qualitative properties are best isolated if orthogonal considerations are left aside. There are two stages. In the first stage, the seller designs an auction and allocates the good accordingly. This stage is similar to the game analyzed in Section 3. In the second stage, buyers decide whether to trade ex post. When resale is an option, bidders may agree on ex post transfers in exchange of the good. Those transfers can be seen as side-payments in the general allocation mechanism.[30]

5.1. The resale problem

Suppose agent i was allocated the good in the auction. For both agents, any payments to the seller are sunk at that stage. If agent $j\neq i$ buys the good from agent i, he increases his profit by $\phi(\theta_j)-\overline{\phi}=v_j$. If agent i keeps the good for himself, his payoff is $\phi(\theta_i)$. If he sells the good to agent j instead, he increases his profit by $v_i=\phi(\theta_i)-\overline{\phi}$. Suppose that the

[26] In the standard literature of signaling, it is possible to fully characterize the separating equilibrium by combining the sequentiality condition $v_0=\text{argmax}\, R(v_0,\alpha(v_0'),\mathcal{A}(v_0'))$ and the boundary condition. When the strategy is a single action, the solution is given by a differential equation.
[27] The formal argument showing that the benchmark optimum is not an equilibrium when v_0 is unknown can be found in the Appendix.
[28] A policy inducing the seller to disclose information before selling has unclear effects. Naturally, a proper model is required to assess those. For a related question, see Daughety and Reinganum (2008) that investigates how firms communicate product quality and choose between voluntary disclosure and signaling through price.
[29] This assumption is made also in Gupta and Lebrun (1999).
[30] Resale is a sort of collusive mechanism in that it improves the welfare of bidders possibly at the expense of the seller. The possibility to collude adds additional incentive constraints and, designing collusion-proof mechanisms requires generally to distort the original second-best mechanism (for an analysis of collusion-proof mechanisms, see for instance Laffont and Martimort, 1997). Similar phenomena arise in the present analysis.

transaction occurs at price t^r. Agent j accepts if t^r is such that $\phi(\theta_j)-t^r \geq \overline{\phi}$. Agent i accepts to trade if t^r is such that $\phi(\theta_i) < \overline{\phi}+t^r$. Overall, trade occurs if it is possible to find a price t^r such that $v_j \geq t^r \geq v_i$. Given valuations are observable, this is true whenever $v_j \geq v_i$ and i was allocated the good in the first place.

Bidders anticipate that resale might take place ex post when the good is not allocated efficiently from their perspective, and they use resale to increase their welfare. In particular, if the seller can observe whether ex post resale takes place and if she can contract upon it, the optimal mechanism consists in allocating the good according to X^* with the same transfers as in $\mathcal{A}^*$, and to impose an extra ex post penalty $p^{**}(v_2)$ on agent 2 if he sells the item to agent 1, such that $p^{**}(v_2) \geq t^r - v_2$ for all t^r and v_2. Avoiding resale is achieved by resorting to an out-of-equilibrium threat and the penalty is such that reselling is not beneficial. If the seller does not observe t^r and v_2, she can always anticipate that agent 1's value is at most $\overline{v}$ and agent 2's value is at least r_2^*. Then any penalty $p^{**} > \overline{v}-r_2^*$ will prevent ex post trade. Overall, as long as the seller can commit to use ex post penalties, it is optimal to do so.[31]

In what follows, we consider the more interesting case where resale cannot be observed by the seller, or cannot be prevented (in particular, ex post penalties cannot be imposed). As before, she offers a mechanism to allocate the item. Once this is done, valuations are disclosed and agents are free to trade. At this stage, we do not impose any rule as to how resale is implemented. We simply assume that when $v_1 > v_2$ and bidder 2 owns the item, trade occurs in which bidder 1 pays $y_1(v_1,v_2) \leq v_1$ and bidder 2 receives $y_2(v_1,v_2) \geq v_2$. Similarly, when $v_1 < v_2$ and bidder 1 owns the item, trade occurs in which bidder 2 pays $x_2(v_1,v_2) \leq v_2$ and bidder 1 receives $x_1(v_1,v_2) \geq v_1$. This formulation allows for different bargaining assumptions between bidders. It also allows for the presence of an intermediary who keeps the difference between the amounts collected and transferred (that is $y_1(v_1,v_2)-y_2(v_1,v_2)$ and $x_2(v_1,v_2)-x_1(v_1,v_2)$). We only require $y_1(v_1,v_2) \geq y_2(v_1,v_2)$ and $x_2(v_1,v_2) \geq x_1(v_1,v_2)$.

Let us denote by $\mathcal{E}_{v_i \in Y_i}$ the 'truncated' expectation over values v_i in subset $Y_i \subset [\underline{v},\overline{v}]$.[32] The utility of bidder 1 with valuation v_1 who reports $\tilde{v}_1$ is now:

$$\begin{aligned} u_1(v_1,\tilde{v}_1) = {} & v_1 \mathcal{E}_{v_2 \in [\underline{v},v_1]}[X_1(\tilde{v}_1,v_2)] + \mathcal{E}_{v_2 \in]v_1,\overline{v}]} ig[x_1(v_1,v_2)X_1(\tilde{v}_1,v_2)] \\ & + \mathcal{E}_{v_2 \in [\underline{v},v_1]}[[v_1 - y_1(v_1,v_2)]X_2(\tilde{v}_1,v_2)] - \alpha(v_0)E_{v_2}X_0(\tilde{v}_1,v_2) - E_{v_2}t_1(\tilde{v}_1,v_2). \end{aligned}$$

Similarly, the utility of bidder 2 with valuation v_2 who reports $\tilde{v}_2$ is

$$\begin{aligned} u_2(v_2,\tilde{v}_2) = {} & v_2 \mathcal{E}_{v_1 \in [\underline{v},v_2]}[X_2(v_1,\tilde{v}_2)] + \mathcal{E}_{v_1 \in]v_2,\overline{v}]}[y_2(v_1,v_2)X_2(v_1,\tilde{v}_2)] \\ & + \mathcal{E}_{v_1 \in [\underline{v},v_2]}[[v_2 - x_2(v_1,v_2)]X_1(v_1,\tilde{v}_2)] - E_{v_1}t_2(v_1,\tilde{v}_2). \end{aligned}$$

Resale may distort the incentives to reveal truthfully, even when information is fully disclosed ex post. This follows from direct inspection of the utilities. If ex post trade is in the hands of a third party (or an intermediary) who extracts the full surplus of resale, that is if $x_i(v_1,v_2) = v_i$ and $y_i(v_1,v_2) = v_i$ for all $i = 1, 2$ the incentives to reveal in the primary auction are not affected. At equilibrium, (i) any time the seller allocates the good efficiently (from the perspective of bidders), resale does not improve upon this situation and (ii) any time the seller allocates it inefficiently, resale results in no surplus. Therefore, the incentive compatibility constraint takes the same form as in Section 3. We will consider this case in Section 5.2. If this is not true, incentive compatibility constraints are affected by the payments $x_i(v_1,v_2)$ and $y_i(v_1,v_2)$. We will analyze one such situation in Section 5.3.

In all cases, the seller anticipates that she will suffer the externality if she sells the good to agent 2 and $v_1 > v_2$: with probability $1-X_0(v)$, the good is allocated and ends up in the hands of agent 1 if $v_1 > v_2$. The optimal direct mechanism solves program $\mathcal{P}^R$:

$$\begin{aligned} \mathcal{P}^R: \ \max R_0^R = & \int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{\overline{v}} [t_1(v) + t_2(v) + v_0\, X_0(v) - \alpha(v_1)(1-X_0(v))1_{v_1 \geq v_2}]\, dF(v_1)\, dF(v_2) \\ \text{s.t. } & u_i(v_i) \geq u_i(v_i,\tilde{v}_i) \quad \forall i, v_i, \tilde{v}_i \\ & u_i(v_i) \geq w_i \quad \forall i, v_i \\ & X_0(v) \geq 0,\ X_1(v) \geq 0,\ X_2(v) \geq 0,\ X_0(v) + X_1(v) + X_2(v) = 1 \quad \forall v. \end{aligned}$$

5.2. Resale with an intermediary

We first consider the case where ex post trade is organized by an intermediary who extracts the full surplus. This assumption is not necessarily realistic but it allows to isolate the externality-changing effect of resale on the original mechanism. As explained earlier, incentive compatibility constraints write as in the first part of the analysis. Using the same methodology as for the proof of Lemma 1, the seller's optimization program becomes $\mathcal{P}^{RI}$

$$\mathcal{P}^{RI}: \ \max \quad \int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{\overline{v}} [X_1(v)\pi_1^{RI}(v) + X_2(v)\pi_2^{RI}(v)]\, dF(v_1)\, dF(v_2) + v_0$$

[31] This rationalizes the behavior of Valencia C.F.

[32] Formally, let $k(\cdot)$ be a function of v_i, we have $\mathcal{E}_{v_i \in Y_i} k(v_i) = \int_{Y_i} k(y) f(v_i)\, dv_i$.

$$\text{s.t.}\quad \begin{array}{ll} E_{v_j}X_i(v_i',v_j)\le E_{v_j}X_i(v_i,v_j)\quad \forall v_i'\le v_i,\quad \forall\{i,j\}=\{1,2\} & \text{(M)}\\ X_0(v)\ge 0,\ X_1(v)\ge 0,\ X_2(v)\ge 0,\ X_0(v)+X_1(v)+X_2(v)=1\quad \forall v & \text{(F)}, \end{array}$$

where $\pi_1^{RI}(v_1,v_2)=v_1-\alpha(v_1)1_{v_1>v_2}+\alpha(v_0)-v_0-(1-F(v_1))/f(v_1)$ and $\pi_2^{RI}(v_1,v_2)=v_2-\alpha(v_1)1_{v_1>v_2}+\alpha(v_0)-v_0-(1-F(v_2))/f(v_2)$. The new virtual surplus reflect the fact that whenever $v_1>v_2$, agent 1 obtains the good and externalities result. When the seller allocates the good to agent 1, she suffers the externality only if he keeps the good. By contrast, when she allocates it to agent 2, she suffers the externality when resale takes place. The optimal mechanism is as follows.

Proposition 5. *The optimal mechanism with resale by an intermediary $\mathcal{A}^{RI}$ entails*:

$$\begin{array}{ll} X_1^{RI}(v)=1 & \text{if } v_1\ge \max(r_1^*,v_2\},\\ X_2^{RI}(v)=1 & \text{if } v_2\ge \max\{r_2^*,v_1\},\\ X_0^{RI}(v)=1 & \text{otherwise}. \end{array}$$

(i) *The agent with highest valuation gets the good.*
(ii) *The reserve prices for agent1 and 2 are r_1^* and r_2^*.*
(iii) *Compared to the situation without resale, the good is allocated more often to bidder 1 ($X_1^{RI}(v)\ge X_1^*(v)$) less often to bidder 2 ($X_2^{RI}(v)\le X_2^*(v)$) and the seller keeps the good more often ($X_0^{RI}(v)\ge X_0^*(v)$).*
(iv) *Resale never takes place on the equilibrium path.*

Proof. See Appendix A.

Agents trade ex post whenever the good is not allocated efficiently from their perspective, and there is no point in allocating the good to agent 2 when $v_1>v_2$. At equilibrium, bidders never trade on the secondary market. However, given the presence of the externality, the seller feels differently about the bidders and it is optimal for her to keep the asymmetric reserve prices r_1^* and r_2^*. Compared to the option of keeping the good, and provided the good is not transferred ex post, the amounts that can be extracted by sending to buyer 1 or buyer 2 are the same as in Proposition 1, and the reserve prices are therefore identical. However, this strategy is optimal only as long as the item is not transferred, and the seller tends to sell the good less often as a commitment device. The allocation is represented Fig. 2.

Given the seller feels differently about the bidders, the possibility of resale acts as a threat: the seller must design a mechanism that reduces this threat. This can be done by allocating the good less often. This results in an increased ex post inefficiency from the perspective of the seller. Also, the seller controls the threat by allocating the good to agent 1 at the correct time. The mechanism is therefore efficient from the perspective of bidders.

5.3. Resale by the winner

We now assume that the bidder who obtains the good has full bargaining power in the resale stage. Contrary to the previous case, rents are now generated ex post. If bidder 1 owns the good and $v_1<v_2$, the good is transferred to bidder 2 and $x_1(v)=x_2(v)=v_2$. Similarly, if bidder 2 owns the good and $v_1>v_2$, the good is transferred to bidder 1 and $y_1(v)=y_2(v)=v_1$. The expected utility of the bidders are therefore:

$$u_1(v_1,\tilde v_1)=v_1\mathcal{E}_{v_2\in[\underline{v},v_1]}[X_1(\tilde v_1,v_2)]+\mathcal{E}_{v_2\in]v_1,\overline{v}]}[v_2X_1(\tilde v_1,v_2)]-\alpha(v_0)E_{v_2}X_0(\tilde v_1,v_2)-E_{v_2}t_1(\tilde v_1,v_2),$$
$$u_2(v_2,\tilde v_2)=v_2\mathcal{E}_{v_1\in[\underline{v},v_2]}[X_2(v_1,\tilde v_2)]+\mathcal{E}_{v_1\in]v_2,\overline{v}]}[v_1X_2(v_1,\tilde v_2)]-E_{v_1}t_2(v_1,\tilde v_2).$$

Let $\pi_1^F(v_1)=v_1-\alpha(v_1)+\alpha(v_0)-v_0$ and $\pi_2^F(v_1)=v_2+\alpha(v_0)-v_0$.

Lemma 3. *The seller's optimization program is equivalent to*:

$$\mathcal{P}^{RW}:\max \int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{v_1}[X_1(v)\pi_1^*(v_1)+X_2(v)\pi_1^F(v_1)]\,dF(v_2)\,dF(v_1)+\int_{\underline{v}}^{\overline{v}}\int_{v_1}^{\overline{v}}[X_1(v)\pi_2^F(v_2)+X_2(v)\pi_2^*(v_2)]\,dF(v_2)\,dF(v_1)+v_0\text{s.t.}$$

$$\begin{array}{l} (v_i-v_i')\mathcal{E}_{v_j\in[\underline{v},v_i']}[X_i(v_i',v_j)]+\mathcal{E}_{v_j\in(v_i',v_i)}[(v_i-v_j)X_i(v_i',v_j)]\\ \quad\le \int_{v_i'}^{v_i}\mathcal{E}_{v_j\in[\underline{v},s]}X_i(s,v_j)\,ds\le (v_i-v_i')\mathcal{E}_{v_j\in[\underline{v},v_i]}[X_i(v_i,v_j)]\\ \quad\quad+\mathcal{E}_{v_j\in(v_i',v_i)}[(v_i'-v_j)X_i(v_i,v_j)]\forall v_i'\le v_i,\ \forall\{i,j\}=\{1,2\}, \qquad (\widehat{\text{IC}})\\ X_0(v)\ge 0,\ X_1(v)\ge 0,\ X_2(v)\ge 0,\ X_0(v)+X_1(v)+X_2(v)=1\quad \forall v, \qquad \text{(F)} \end{array}$$

Proof. See Appendix A.

The owner of the good extracts the full surplus from trade in the resale stage. Then, each bidder knows that he will have no rent ex post any time his valuation is high and his lower valuation rival obtains the object. Then, the new virtual surplus have an intuitive interpretation. Suppose $v_2 < v_1$. If the seller allocates the good to bidder 1, there will not be resale. The problem is similar to the one we analyzed in Section 3 and the virtual surplus is the same. If the seller allocates the good to bidder 2, it will change hands ex post. The (revised) value of bidder 2 is therefore v_1, and the seller can extract that amount from him. Besides, she can also extract $\alpha(v_0)$ from bidder 1 who will not suffer the externality. Overall, she can obtain the amount she would extract under full information by selling to bidder 1 directly. Given the good will be transferred ex post to bidder 1, the seller values the option of keeping the good at $v_0+\alpha(v_1)$. The virtual surplus is the same as if information were complete, $\pi_1^F(v_1)$. Last $(\widehat{\text{IC}})$ guarantees that telling the truth is a global optimum.

Proposition 6. *The optimal mechanism with resale by the owner $\mathcal{A}^{\mathcal{RW}}$ entails*:

$$\begin{aligned}
X_1^{RW}(v) &= q(v) && \text{if } v_1 \geq \max\{v_2; r_1^*\}\\
&= 1-q(v) && \text{if } v_1 \leq v_2 \text{ and } v_2 \geq r_2^F,\\
X_2^{RW}(v) &= q(v) && \text{if } v_2 \geq \max\{v_1; r_2^*\}\\
&= 1-q(v) && \text{if } v_2 \leq v_1 \text{ and } v_1 \geq r_1^F,\\
X_0^{RW}(v) &= 1 && \text{otherwise},
\end{aligned}$$

where $q(v)$ is a probability satisfying $\mathcal{E}_{v_j\in[\underline{v},v_i']}q(v_i',v_j) \leq \mathcal{E}_{v_j\in[\underline{v},v_i]}q(v_i,v_j)$ for all $r_i^* \leq v_i' \leq v_i$.

(i) *The agent with lowest valuation gets the good with a positive probability.*
(ii) *Compared to the situation without resale, the seller keeps the good more often than under full information ($X_0^{RW}(v) \geq X_0^F(v)$); and less often than when an intermediary extracts ex post rents ($X_0^{RW}(v) \leq X_0^{RI}(v)$).*
(iii) *Resale sometimes takes place on the equilibrium path.*

Proof. See Appendix A.

The seller could take advantage of the resale problem in the following way. She knows that any inefficient trade will be followed by resale and the full value will be extracted. Therefore, provided agents report truthfully, she prefers to allocate the good to the lower valuation and extract the surplus he intends to extract in the resale market. In other words, it is optimal for her to post the reserve prices r_1^F and r_2^F and allocate the good to agent v_j when $v_j < v_i$ and $v_i > r_i^F$. However, this strategy is not incentive compatible: a high type anticipates that he will not be allocated the good in the auction and will be charged his valuation in the resale market. Therefore, he strictly prefers to misrepresent his type.

If the seller decides to sell the good to the highest valuation instead, then it is optimal to adopt the same mechanism as in the previous section (see Fig. 2). This mechanism is incentive compatible but generates a lower revenue. The seller can increase her payoff by randomizing between these two mechanisms (point (i)): she gets her preferred allocation only as long as it does not hurt the incentives to report. At equilibrium, given $r_1^* > r_1^F$ and $r_2^* > r_2^F$, the good is allocated more often than in the previous section, but less often than under complete information (point (ii)). The equilibrium allocation in the mechanism as well as the final allocation are represented in Fig. 3.

Ex post trade occurs with a positive probability (point (iii)). The seller can benefit from ex post trade by extracting part of the rents that will be generated ex post. This was not true in the previous section because no rents were left to the agents. We conjecture that this effect should remain as long as at least one bidder gets a positive surplus from trading in the resale market. In particular, this should apply to the more general case where valuations are not public knowledge ex

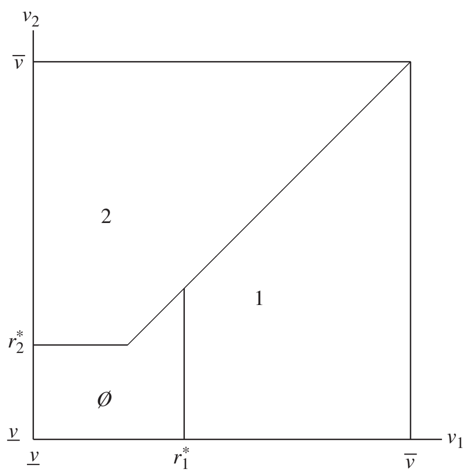

Fig. 2. Optimal mechanism with resale by an intermediary.

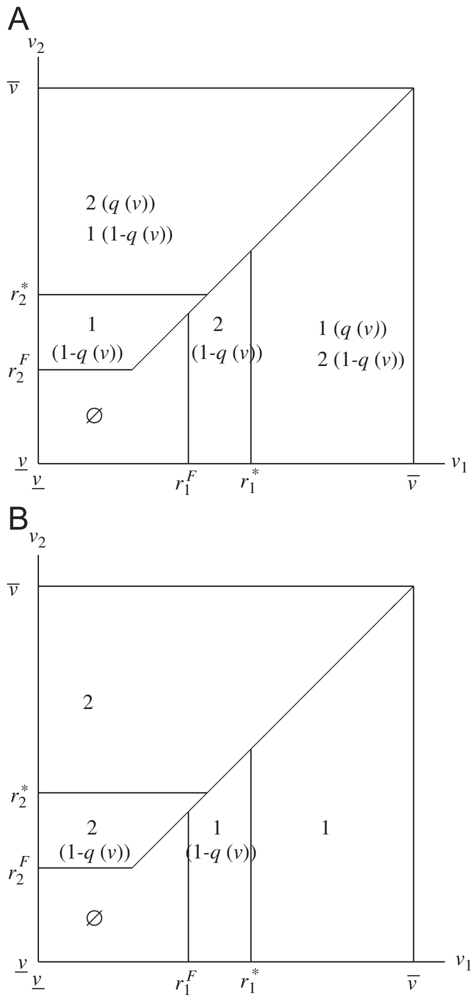

Fig. 3. (a) Optimal mechanism with resale by the winner and (b) final allocation with resale by the winner.

post.[33] However, the interaction between the qualitative effects generated by the presence of asymmetric information in the resale market and the qualitative properties generated by the resale format itself is out of the scope of the present paper. We shall also note that the optimal mechanism could be replicated by a resale-proof mechanism that incorporates the intra-bidder payments of the resale stage. Suppose that, in $\mathcal{A}^{\mathcal{RW}}$, agent i is allocated the item, $T_k^{RW}(v)$ $k=1,2$ are the ex post transfers and resale follows. Then the seller could allocate the good to j and charge $\tilde{T}_i(v)=T_i^{RW}(v)+v_j$ and $\tilde{T}_j(v)=T_j^{RW}(v)-v_j$. In this resale-proof mechanism, agent i receives the good only with some probability when $v_i \in (r_i^F,r_i^*)$ to replicate the final allocation in the optimal (non-resale-proof) mechanism.

The possibility of resale induces the seller to distort the allocation and the way bidders organize ex post trade affects her incentives. Even though the seller cannot influence the modalities of ex post trade, she can allocate the item to prevent resale. She can also attempt to capture part of the rents that will be generated ex post by charging the winner a share of what he may collect through resale. If the rents generated in the resale stage do not accrue to bidders, then she can use only the first tool. By contrast, if the winner can obtain rents through ex post trade, then both tools are available. There is a motive for charging part of the extra benefit the lowest valuation agent obtains through resale.

6. Conclusion

In this paper, we have focused on the case where a seller decides whether to allocate an item to a competitor or a non-competitor or to keep it for herself. We have shown that the optimal allocation rule favors the buyers who do not interact

[33] Also, some efficient trades will not be undertaken because of asymmetric information between bidders.

ex post with the seller. This is true in all the three variants we have analyzed. In particular, the bidder with the highest valuation does not necessarily obtain the good. This is the case in the benchmark static case, in the signaling case but also sometimes when resale is an option. Also, the solution under double asymmetric information may exhibit more or less inefficiencies than the standard simple asymmetric information benchmark. The difficulty of the seller to signal her type will result in either a reduction or an increase in the probability of keeping the good. Therefore, a policy aiming at inducing the seller to disclose information before selling has unclear effects.

The results obtained in the presence of resale have nice properties that may be interesting to analyze further within a more general model. The idea that the resale format should affect crucially the initial mechanism is intuitive. We have isolated a few reasons why, still the analysis does not provide a complete understanding. It is important to note however that technical difficulties emerge with almost all resale formats (to deal in particular with incentive compatibility) which may place a bound on the results to obtain in a general setting. The most natural extension is perhaps to restore incomplete information in the resale stage. The question here is what type of beliefs are buyers left with after the primary allocation and how can the seller design a mechanism capable of shaping the beliefs to her advantage. Intuitively, some ex post efficient trades should not be undertaken in this extended model, which would benefit the seller. Also, in our particular setting, the seller could a priori be induced to participate in the resale auction as she still wants to avoid the good to be transferred to her rival. If such possibility is allowed, the resale problem becomes an interesting asymmetric three-person bargaining problem. Our analysis could also be extended to that case.

It may be interesting to extend the analysis to the case where the seller can keep a ‘copy’ of the asset even if she sells to other bidders. This would allow to tailor the analysis to the licensing problem.[34] In that case, selling a license does not prevent the patentee to use her innovation. Even though the licensing strategy should have similarities with the case analyzed here (namely, it is optimal to treat potential competitors asymmetrically), it should also reflect the fact that the patentee never loses her property right. Examples of this situation abound and strategies differ largely. Microsoft for instance licenses the use of its application software (such as Word or Excel) for use on competing operating systems (such as Apple’s). Others, like Apple, refuse categorically to license their innovations (such as the MAC operating system, the iPod or the iPhone). This strategy has proved successful in Apple’s case but has not for Sony with Betamax.

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Appendix A

Proof of Lemma 1. The utility of agent 1 is $u_1(v_1,v_1')=E_{v_2}[v_1X_1(v_1',v_2)-\alpha(v_0)X_0(v_1',v_2)-t_1(v_1',v_2)]$. Note that $u_1(v_1,v_1')=u_1(v_1',v_1')+(v_1-v_1')E_{v_2}X_1(v_1',v_2)$, then the incentive compatibility constraint is equivalent to:

$$u_1(v_1,v_1)\geq u_1(v_1',v_1')+(v_1-v_1')E_{v_2}X_1(v_1',v_2). \tag{5}$$

Using this inequality twice, we have

$$(v_1-v_1')E_{v_2}X_1(v_1',v_2)\leq u_1(v_1,v_1)-u_1(v_1',v_1')\leq(v_1-v_1')E_{v_2}X_1(v_1,v_2). \tag{6}$$

Agent 1 reveals truthfully if the following necessary condition is satisfied:

$$E_{v_2}X_1(v_1',v_2)\leq E_{v_2}X_1(v_1,v_2)\quad\forall v_1'\leq v_1. \tag{7}$$

(6) must hold for all v_1' and all $v_1=v_1'+\delta$ with $\delta>0$. Since $E_{v_2}X_1(v_1,v_2)$ is increasing in v_1, we can take the Riemann integral. Then, the agent reveals truthfully if we also have

$$u_1(v_1)-u_1(v_1')=\int_{v_1'}^{v_1}E_{v_2}X_1(s,v_2)\,ds\quad\forall v_1'\leq v_1. \tag{8}$$

To complete the proof, we need to verify that (8) and (7) imply (14). Suppose $v_1'\leq v_1$, then given (8) and (7) , we have

$$\begin{aligned}u_1(v_1,v_1)&=u_1(v_1',v_1')+\int_{v_1'}^{v_1}E_{v_2}X_1(s,v_2)\,ds\\&\geq u_1(v_1',v_1')+\int_{v_1'}^{v_1}E_{v_2}X_1(v_1',v_2)\,ds\\&=u_1(v_1',v_1')+(v_1-v_1')E_{v_2}X_1(v_1',v_2).\end{aligned}$$

The utility of agent 2 is $u_2(v_2,v_2')=E_{v_1}[v_2X_2(v_1,v_2')-t_2(v_1,v_2')]$. We have $u_2(v_2,v_2')=u_2(v_2',v_2')+(v_2-v_2')E_{v_1}X_2(v_2',v_1)$, then

[34] For a recent review of the literature on licensing, see Scotchmer (2004).

using the same arguments as before, the agent reveals truthfully if and only if

$$E_{v_1}X_2(v_2',v_1) \le E_{v_1}X_2(v_2,v_1) \quad \forall v_2' \le v_2. \tag{9}$$

$$u_2(v_2)-u_2(v_2') = \int_{v_2'}^{v_2} E_{v_1}X_2(s,v_1)\,ds \quad \forall v_2' \le v_2. \tag{10}$$

The seller maximizes her expected revenue under constraints (8), (7), (10), (9) (to induce truthful revelation) and the remaining constraints (IR1), (IR2) and (F).[35] The expected revenue of the seller is

$$E_{v_1,v_2}[[1-X_1(v)-X_2(v)]v_0-\alpha(v_1)X_1(v)+t_1(v)+t_2(v)].$$

The expected transfers paid by agents 1 and 2, respectively, are

$$E_{v_2}t_1(v) = E_{v_2}[v_1X_1(v)-\alpha(v_0)X_0(v)]-u_1(v_1),$$

$$E_{v_1}t_2(v) = E_{v_1}v_2X_2(v)-u_2(v_2).$$

Using (8) and (10), the expected utility of agents 1 and 2, respectively are

$$u_1(v_1) = \int_{\underline{v}}^{v_1} E_{v_2}X_1(s,v_2)\,ds+u_1(\underline{v}),$$

$$u_2(v_2) = \int_{\underline{v}}^{v_2} E_{v_1}X_2(s,v_1)\,ds+u_2(\underline{v}).$$

The seller does not want to give extra rents and the individual rationality constraint of each agent is binding in $\underline{v}$, i.e. the optimal mechanism is such that $u_1(\underline{v}) = w_1 = -\alpha(v_0)$ and $u_2(\underline{v}) = w_2 = 0$. Replacing the equilibrium expressions of the expected transfers in the expected revenue and integrating by parts, the objective of the seller is to maximize

$$\int_{v_1}\int_{v_2}\left[X_1(v)\left[v_1-\frac{1-F(v_1)}{f(v_1)}+\alpha(v_0)-\alpha(v_1)-v_0\right]+X_2(v)\left[v_2-\frac{1-F(v_2)}{f(v_2)}+\alpha(v_0)-v_0\right]\right]dF(v_1)\,dF(v_2)+v_0,$$

under (7), (9), (F_0) and (F_1). The virtual surplus are $\pi_1^*(v_1) = v_1-(1-F(v_1))/f(v_1)+\alpha(v_0)-\alpha(v_1)-v_0$ and $\pi_2^*(v_2) = v_2-(1-F(v_2))/f(v_2)+\alpha(v_0)-v_0$. □

Proof of Proposition 1. Given Assumption 1, the virtual surplus $\pi_1^*(v_1)$ and $\pi_2^*(v_2)$ are increasing in v_1 and v_2, respectively. Neglecting the constraints, the mechanism that maximizes the seller's expected revenue is

$$\begin{aligned} &X_1(v)=1 \quad \text{if } \pi_1^*(v_1)>0 \quad \text{and} \quad \pi_1^*(v_1)>\pi_2^*(v_2),\\ &X_2(v)=1 \quad \text{if } \pi_2(v_2)>0 \quad \text{and} \quad \pi_2^*(v_2)>\pi_1^*(v_2),\\ &X_0(v)=1 \quad \text{otherwise}.\end{aligned}$$

Consider r_1^* and r_2^* defined in Proposition 1. We have $r_1^* > r_2^*$. For all $v_1 > v_1^{\min}$, we have $\pi_1^*(v_1) = \pi_2^*(h(v_1))$. We can check easily (by differentiating the previous equation) that $h(v_1)$ is increasing in v_1 and $h(v_1) < v_1$. Naturally, we also have $h(r_1^*) = r_2^*$ (by construction) for all $r_1^* > v_1^{\min}$. Then, the mechanism that maximizes the seller's revenue is simply:

$$\begin{aligned} &X_1(v)=1 \quad \text{if } v_1 \ge \max(r_1^*,h^{-1}(v_2)\},\\ &X_2(v)=1 \quad \text{if } v_2 \ge \max\{r_2^*,h(v_1)\},\\ &X_0(v)=1 \quad \text{otherwise}.\end{aligned}$$

with $h^{-1}(y) = \overline{v}$ for all $y > y^*$, where $y^* = h^{-1}(\overline{v})$. This mechanism satisfies (F_0) and (F_1). If $v_1 < r_1^*$, then $E_{v_2}X_1(v) = 0$. If $v_1 > r_1^*$, $EX_1(v) = F(h(v_1))$ which is increasing in v_1, and (12) is satisfied. Similarly, when $v_2 < r_2^*$, $E_{v_1}X_2(v) = 0$ and when $v_2 > r_2^*$, $E_{v_1}X_2(v) = F(h^{-1}(v_2))$. Then, the probability that agent 2 gets the good increases and (9) is also satisfied. The expected transfers are

$$E_{v_2}t_1^*(v) = E_{v_2}[v_1X_1^*(v)-\alpha(v_0)X_0^*(v)]-\int_{\underline{v}}^{v_1} E_{v_2}X_1^*(s,v_2)\,ds+\alpha v_0.$$

$$E_{v_1}t_2^*(v) = E_{v_1}v_2X_2^*(v)-\int_{\underline{v}}^{v_2} E_{v_1}X_2^*(s,v_1)\,ds.$$

Under complete information, the seller can extract rents from both agents. The surplus derived from selling to bidders 1 and 2 are, respectively, $\pi_1^F(v_1) = v_1+\alpha(v_0)-v_0-\alpha(v_1)$ and $\pi_2^F(v_2) = v_2+\alpha(v_0)-v_0$. Let us denote by $X_i^F(v)$ the allocation rule

[35] Note that the proof is similar to Myerson (1981).

to bidder i under complete information.

$$X_1^F(v)=1 \quad \text{if } v_1 \geq \max(r_1^F, z^{-1}(v_2)\},$$
$$X_2^F(v)=1 \quad \text{if } v_2 \geq \max\{r_2^F, z(v_1)\},$$
$$X_0^F(v)=1 \quad \text{otherwise},$$

where r_1^F and r_2^F are the reserve prices faced by agents 1 and 2, respectively. They are such that $\pi_1^F(r_1^F)=0$ and $\pi_2^F(r_2^F)=0$, respectively; also $z(v_1)=\min\{v_2 \,|\, \pi_1^F(v_1) \leq \pi_2^F(z(v_1))\}$, that is $z(v_1)=v_1-\alpha(v_1)$. We have $r_1^F \neq r_2^F$ and $z(v_1)<v_1$. By direct inspection of the surplus under complete information and virtual surplus under incomplete information, we have that $r_1^F<r_1^*$ and $r_2^F<r_2^*$. Besides, $z(v_1)<h(v_1)$. This proves Proposition 1. □

Proof of Proposition 2. Suppose a seller with type v_0 mimics a seller with type v_0', then

$$R(v_0,\gamma(\mathcal{A}(v_0')),\mathcal{A}(v_0'))=\int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{\overline{v}}[X_1(v;v_0')\tilde{\pi}_1(v_1;v_0,\gamma(\mathcal{A}(v_0')))+X_2(v;v_0')\tilde{\pi}_2(v_2;v_0,\gamma(\mathcal{A}(v_0')))]\,dF(v_1)\,dF(v_2)+v_0$$

and we have

$$R(v_0,\gamma(\mathcal{A}(v_0')),\mathcal{A}(v_0'))=R(v_0',\gamma(\mathcal{A}(v_0')),\mathcal{A}(v_0'))+(v_0-v_0')\int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{\overline{v}}[1-X_1(v;v_0')-X_2(v;v_0')]\,dF(v_1)\,dF(v_2).$$

The sequential rationality condition can be rewritten as

$$R(v_0,\gamma(\mathcal{A}(v_0)),\mathcal{A}(v_0))-R(v_0',\gamma(\mathcal{A}(v_0')),\mathcal{A}(v_0'))\geq(v_0-v_0')\int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{\overline{v}}[1-X_1(v;v_0')-X_2(v;v_0')]\,dF(v_1)\,dF(v_2).$$

Applying this inequality twice, we also have

$$R(v_0,\gamma(\mathcal{A}(v_0)),\mathcal{A}(v_0))-R(v_0',\gamma(\mathcal{A}(v_0')),\mathcal{A}(v_0'))\leq(v_0-v_0')\int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{\overline{v}}[1-X_1(v;v_0)-X_2(v;v_0)]\,dF(v_1)\,dF(v_2).$$

To be sequentially rational, the mechanism must be such that for all $v_0 \geq v_0'$:

$$\int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{\overline{v}}[1-X_1(v;v_0')-X_2(v;v_0')]\,dF(v_1)\,dF(v_2)\leq\int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{\overline{v}}[1-X_1(v;v_0)-X_2(v;v_0)]\,dF(v_1)\,dF(v_2).$$

The probability of keeping the good must be non-decreasing in the type of the seller. This proves (ii).

When v_0 increases, it becomes less profitable to sell the object. Provided this is true, we can take the Riemann integral and

$$R(v_0,\gamma(\mathcal{A}(v_0)),\mathcal{A}(v_0))-R(v_0',\gamma(\mathcal{A}(v_0')),\mathcal{A}(v_0'))=\int_{v_0'}^{v_0}\int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{\overline{v}}[1-X_1(v;s)-X_2(v;s)]\,dF(v_1)\,dF(v_2)\,ds.$$

The revenue from the auction is therefore increasing in the type of the seller. This proves (i).

Given the probability of keeping the good is non-decreasing in the type of the seller, a higher probability of selling the good signals a lower type. When $\alpha'>0$, this signals a lower externality, and when $\alpha'<0$, this signals a higher externality. This proves (iii).

Under complete information about the seller's type, the equilibrium revenue is

$$\int_{\underline{v}}^{r_1^*}\int_{r_2^*}^{\overline{v}}\pi_2^*(v_2)\,dF(v_2)\,dF(v_1)+\int_{r_1^*}^{\overline{v}}\int_{\underline{v}}^{h(v_1)}\pi_1^*(v_1)\,dF(v_2)\,dF(v_1)+\int_{r_1^*}^{\overline{v}}\int_{h(v_1)}^{\overline{v}}\pi_2^*(v_2)\,dF(v_2)\,dF(v_1)+v_0$$
$$=F(r_1^*(v_0))\int_{r_2^*(v_0)}^{\overline{v}}\pi_2^*(v_2,v_0)\,dF(v_2)+\int_{r_1^*(v_0)}^{\overline{v}}F(h(v_1))\pi_1^*(v_1,v_0)\,dF(v_1)+\int_{r_1^*(v_0)}^{\overline{v}}\int_{h(v_1)}^{\overline{v}}\pi_2^*(v_2,v_0)\,dF(v_2)\,dF(v_1)+v_0.$$

The derivative of this expression is

$$[\alpha'(v_0)-1][1-F(r_1^*)F(r_2^*)]+1.$$

We also have that

$$\int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{\overline{v}}[1-X_1(v;v_0)-X_2(v;v_0)]\,dF(v_1)\,dF(v_2)=F(r_1^*)F(r_2^*).$$

Therefore, the allocation under complete information is not sequentially rational unless $\alpha'(v_0)=0$ for all v_0. This corresponds to the standard Independent Private Value (IPV) model. When $\alpha'>0$, the equilibrium revenue increases too fast, and when $\alpha'<0$, the equilibrium revenue increases too slowly. This proves the last claim. □

Proof of Proposition 3. An incentive compatible separating equilibrium is such that for all $v_0 > v_0'$

$$R(v_0,\alpha(v_0),\mathcal{A}(v_0)) - R(v_0',\alpha(v_0'),\mathcal{A}(v_0')) = \int_{v_0'}^{v_0}\int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{\overline{v}} [X_0(v;s)]\, dF(v_1)\, dF(v_2)\, ds.$$

This is the counterpart of condition (i) in the last proposition. To be consistent with an equilibrium, the mechanism must satisfy a boundary condition. Let $\mathcal{A}^*(v_0)$ be the optimal auction mechanism offered by the seller when her type is known. For all possible beliefs, we have $R_2(v_0,\gamma,\mathcal{A}) \geq 0$. Consider now an equilibrium belief function $\hat{\gamma}$ and assume $\alpha'(\cdot) \geq 0$. We have

$$\begin{aligned} R(\underline{v},\hat{\gamma}(\mathcal{A}^*(\underline{v})),\mathcal{A}^*(\underline{v})) &\geq R(\underline{v},\alpha(\underline{v}),\mathcal{A}^*(\underline{v}))\\ &> R(\underline{v},\alpha(\underline{v}),\mathcal{A}(\underline{v}))\\ &= R(\underline{v},\hat{\gamma}(\mathcal{A}(\underline{v})),\mathcal{A}(\underline{v})).\end{aligned}$$

This violates sequential rationality unless $\mathcal{A}(\underline{v}) = \mathcal{A}^*(\underline{v})$. Assume now $\alpha'(\cdot) \leq 0$. We have

$$\begin{aligned} R(\overline{v},\hat{\gamma}(\mathcal{A}^*(\overline{v})),\mathcal{A}^*(\overline{v})) &\geq R(\overline{v},\alpha(\overline{v}),\mathcal{A}^*(\overline{v}))\\ &> R(\overline{v},\alpha(\overline{v}),\mathcal{A}(\overline{v}))\\ &= R(\overline{v},\hat{\gamma}(\mathcal{A}(\overline{v})),\mathcal{A}(\overline{v})).\end{aligned}$$

This violates sequential rationality unless $\mathcal{A}(\overline{v}) = \mathcal{A}^*(\overline{v})$. Overall, when $\alpha'(\cdot) > 0$, we have

$$R(v_0,\alpha(v_0),\mathcal{A}(v_0)) = \int_{\underline{v}}^{v_0}\int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{\overline{v}} [X_0(v;s)]\, dF(v_1)\, dF(v_2)\, ds + R(\underline{v},\alpha(\underline{v}),\mathcal{A}^*(\underline{v}))$$

and when $\alpha'(\cdot) < 0$, we have

$$R(v_0,\alpha(v_0),\mathcal{A}(v_0)) = R(\overline{v},\alpha(\overline{v}),\mathcal{A}^*(\overline{v})) - \int_{v_0}^{\overline{v}}\int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{\overline{v}} [X_0(v;s)]\, dF(v_1)\, dF(v_2)\, ds.$$

This leads to the result. □

Proof of Proposition 4. First, for any belief agents may hold, the virtual surplus $\tilde{\pi}_1(v_1) \gtrless \tilde{\pi}_2(v_1)$ if $v_2 \lesseqgtr h(v_1)$. Then, among all mechanisms that are sequentially rational, it is optimal to pick a mechanism such that, whenever the good is allocated, it is allocated to bidder 2 if $v_2 \geq h(v_1)$.

Second, in any mechanism that satisfies the monotonicity condition necessary to induce agent *i* to reveal truthfully, and provided (F) is satisfied, there exists a value $r_i(v_0)$ such that $E_{v_j}X_i(v_i,v_j) = 0$ if $v_i < r_i(v_0)$ and $E_{v_j}X_i(v_i,v_j) > 0$ if $v_i \geq r_i(v_0)$. In other words, $r_i(v_0)$ is the minimum value to be granted a positive probability of obtaining the good. Consider an allocation rule $\tilde{\mathcal{A}}(v_0)$ such that

$$\begin{aligned} &X_1(v;v_0) = p_1(v;v_0) && \text{if } v_1 \geq \max(r_1(v_0),h^{-1}(v_2)\},\\ &X_2(v;v_0) = p_2(v;v_0) && \text{if } v_2 \geq \max\{r_2(v_0),h(v_1)\},\\ &X_0(v) = 1 && \text{otherwise,}\end{aligned}$$

where $p_i(v;v_0)$ for all v are probabilities such that $\int_{\underline{v}}^{v_i} p_i(v;v_0)f(v_j)\, dv_j \geq \int_{\underline{v}}^{v_i'} p_i(v;v_0)f(v_j)\, dv_j$ for all $v_i > v_i'$ (to guarantee the probability of receiving the good is non-decreasing in the valuation). This mechanism satisfies (F) as well as the monotonicity conditions for both agents and, conditional on selling the good, it allocates it to each agent at the right time. Allocations $\tilde{\mathcal{A}}(v_0)$ are obtained by moving $r_1(v_0)$ and $r_2(v_0)$ along $h(v_1)$. Note that under complete information about v_0, the optimal mechanism consists in setting $r_1(v_0) = r_1^*(v_0)$, $r_2(v_0) = r_2^*(v_0)$ and $p_i(v;v_0) = 1$. When v_0 is unknown, this allocation must be distorted to make sure the revenue satisfies the conditions in Proposition 1 and the probability of keeping the good satisfies condition (ii) in Proposition 2.

We will now show that a solution satisfying these properties exist. Consider the subclass of mechanisms $\tilde{\mathcal{A}}'(v_0)$ such that $p_i(v;v_0) = 1$ for all *i* and such that $r_2(v_0) = h(r_1(v_0))$. The problem now consists in choosing $r_1(v_0)$ to satisfy the conditions in Proposition 1 and condition (ii) in Proposition 2. This problem is reminiscent of the problem analyzed by Jullien and Mariotti (2006). It can be solved following the procedure developed in Mailath (1987) provided we show that the expected revenue satisfies five conditions. The expected revenue associated with the mechanism we restrict to when the seller type is v_0, her perceived type is v_0' and her reserve price decision is r_1 writes as

$$\begin{aligned}\tilde{R}(v_0,v_0',r_1) = &\int_{\underline{v}}^{r_1}\int_{h(r_1)}^{\overline{v}} \hat{\pi}_2(v_2;v_0',v_0)\, dF(v_2)\, dF(v_1) + \int_{r_1}^{\overline{v}}\int_{\underline{v}}^{h(v_1)} \hat{\pi}_1(v_1;v_0',v_0)\, dF(v_2)\, dF(v_1)\\ &+ \int_{r_1}^{\overline{v}}\int_{h(v_1)}^{\overline{v}} \hat{\pi}_2(v_2;v_0',v_0)\, dF(v_2)\, dF(v_1) + v_0 = F(r_1)\int_{h(r_1)}^{\overline{v}} \hat{\pi}_2(v_2;v_0',v_0)\, dF(v_2)\\ &+ \int_{r_1}^{\overline{v}} F(h(v_1))\hat{\pi}_1(v_1;v_0',v_0)\, dF(v_1) + \int_{r_1}^{\overline{v}}\int_{h(v_1)}^{\overline{v}} \hat{\pi}_2(v_2;v_0',v_0)\, dF(v_2)\, dF(v_1) + v_0,\end{aligned}$$

$\hat{\pi}_1(v_1;v_0',v_0) = v_1 - \alpha(v_1) + \alpha(v_0') - v_0 - (1-F(v_1))/f(v_1)$ and $\hat{\pi}_2(v_2;v_0',v_0) = v_2 + \alpha(v_0') - v_0 - (1-F(v_2))/f(v_2)$.
For all $v_0 \in [\underline{v},\overline{v}]$, all $v_0' \in [\underline{v},\overline{v}]$ and all $r_1 \geq v_1^{\min}$, we have

(1) $\tilde{R}(v_0,v'_0,r_1)$ is C^2 on $[\underline{v},\overline{v}]^2 \times \mathbb{R}$ (given the differentiability of the functions involved).
(2) $\tilde{R}_2(v_0,v'_0,r_1) = \alpha'(v_0)[(1-F(h(r_1))+(1-F(r_1))] \neq 0$ if $\alpha'(\cdot) \neq 0$.
(3) $\tilde{R}_{13}(v_0,v'_0,r_1) = F(r_1)(dh/dr_1)f(h(r_1))+F(h(r_1))f(r_1) \neq 0$.
(4) $\tilde{R}_3(v_0,v_0,r_1)=0$ has the unique solution $r_1^*(v_0)$ and $R_{33}(v_0,v_0,r_1^*(v_0))<0$ (this is the solution obtained in Section 3).
(5) $\tilde{R}_3(v_0,v'_0,r_1)/\tilde{R}_2(v_0,v'_0,r_1)$ is monotonic in v_0 (because $(d/dv_0)[\tilde{R}_3(v_0,v'_0,r_1)/\tilde{R}_2(v_0,v'_0,r_1)] \propto \alpha'(v_0)[(1-F(h(r_1))+ (1-F(r_1))]$ $[F(r_1)(dh/dr_1)f(h(r_1))+F(h(r_1))f(r_1)]$).

Besides, the allocation needs to satisfy the initial condition $r_1^S(\underline{v})=r_1^*(\underline{v})$ if $\alpha'>0$ and $r_1^S(\overline{v})=r_1^*(\overline{v})$ if $\alpha'<0$. Following Mailath (1987), there exists a differentiable incentive compatible separating allocation $r_1^S(v_0)$ such that

$$\frac{dr_1^S}{dv_0} = -\frac{\tilde{R}_2(v_0,v_0,r_1^S(v_0))}{\tilde{R}_3(v_0,v_0,r_1^S(v_0))} = \frac{\alpha'(v_0)[1-F(r_1^S(v_0))F(h(r_1^S(v_0)))]}{\pi_1^*(r_1^S(v_0);v_0)\dfrac{d}{dv_0}(F(r_1^S(v_0))F(h(r_1^S(v_0))))}.$$

(Note that this condition corresponds to condition (i) in Proposition 2, $(d/dv_0)R(v_0,v_0,r_1^S(v_0)) = E_v[X_0(v;v_0)]$.) The solution is monotonic and dr_1^S/dv_0 has the same sign as $R_{13}(v_0,v'_0,r_1)$ (Mailath (1987), Theorem 2). In our case $dr_1^S/dv_0>0$ and therefore condition (ii) in Proposition 2 is satisfied. By inspection of the differential equation, it is easy to see that at equilibrium, the solution is such that the sign of $\pi_1^*(r_1^S(v_0);v_0)$ is the same as the sign of α'. Therefore, when $\alpha'>0$, $r_1^S(v_0)>r_1^*(v_0)$ and when $\alpha'<0$, $r_1^S(v_0)<r_1^*(v_0)$.

The allocation is an equilibrium if we specify appropriate out-of-equilibrium beliefs. For any $\mathcal{A} \notin (\mathcal{A}^S(v_0))_{v_0\in[\underline{v},\overline{v}]}$, let:

$$\gamma(\mathcal{A}) = \begin{cases} \alpha(\underline{v}) & \text{when } \alpha'>0, \\ \alpha(\overline{v}) & \text{when } \alpha'<0. \end{cases}$$

A seller with type v_0 does not have incentives to deviate because agents hold the most pessimistic beliefs when she does not offer a reserve price consistent with the equilibrium profile $r_1^S(v_0)$. When $\alpha'>0$, for any offered mechanism $\mathcal{A}$ entailing the reserve price r_1, agents believe that $\gamma(\mathcal{A})=\alpha(\underline{v})$ when $r_1 \notin [r_1^S(\underline{v}),r_1^S(\overline{v})]$ and $\gamma(\mathcal{A})=\alpha(v_0)$ when $r_1=r_1^S(v_0)$.

$$\tilde{R}_3(v_0,\underline{v},r_1) = -F(r_1)\hat{\pi}_2(h(r_1);\underline{v},v_0)\frac{dh}{dr_1}f(h(r_1)) - F(h(r_1))\hat{\pi}_1(r_1;\underline{v},v_0).$$

Note that $\tilde{R}(v_0,\underline{v},r_1)$ is increasing in r_1 when $r_1<r_1^S(\underline{v})=r_1^*(\underline{v})$. Therefore, for all $r_1<r_1^S(\underline{v})$ we have $R(v_0,\alpha(v_0),\mathcal{A}^S(v_0)) \geq R(v_0,\alpha(\underline{v}),\mathcal{A}^S(\underline{v})) = \tilde{R}(v_0,\underline{v},r_1^S(\underline{v})) > \tilde{R}(v_0,\underline{v},r_1)$. Note also that $\tilde{R}(v_0,\underline{v},r_1)$ is decreasing in r_1 when $r_1>r_1^S(v_0)$,and it is increasing in the second argument (see above condition (2)). Therefore, for all $r_1>r_1^S(\overline{v})(>r_1^S(v_0))$, we have $\tilde{R}(v_0,\underline{v},r_1)<\tilde{R}(v_0,\underline{v},r_1^S(v_0))<\tilde{R}(v_0,v_0,r_1^S(v_0))$. The argument is similar for $\alpha'<0$. □

Proof of Proposition 5. We have $\pi_1^{RI}(v_1,v_2)=v_1-\alpha(v_1)1_{v_1>v_2}+\alpha(v_0)-v_0-(1-F(v_1))/f(v_1)$ and $\pi_2^{RI}(v_1,v_2)=v_2-\alpha(v_1)$ $1_{v_1>v_2}+\alpha(v_0)-v_0-(1-F(v_2))/f(v_2)$, therefore $\pi_1^{RI}(v_1,v_2) \gtrless \pi_2^{RI}(v_1,v_2)$ if $v_1 \gtrless v_2$. If $v_1>v_2$ it is best to sell to 1 provided $\pi_1^{RI}(v_1,v_2)\geq 0$, that is $v_1 \geq r_1^*$. If $v_1<v_2$ it is best to sell to 2 provided $\pi_2^{RI}(v_1,v_2)\geq 0$, that is $v_2 \geq r_2^*$. □

Proof of Lemma 3. We have now

$$u_1(v_1,v'_1) = u_1(v'_1,v'_1)+v_1\mathcal{E}_{v_2\in[\underline{v},v_1]}[X_1(v'_1,v_2)]+\mathcal{E}_{v_2\in(v_1,\overline{v}]}[v_2X_1(v'_1,v_2)]$$
$$-v'_1\mathcal{E}_{v_2\in[\underline{v},v'_1]}[X_1(v'_1,v_2)]-\mathcal{E}_{v_2\in(v'_1,\overline{v}]}[v_2X_1(v'_1,v_2)],$$

then to satisfy incentive compatibility, we must have for all $v_1 \geq v'_1$:

$$u_1(v_1,v_1) \geq u_1(v'_1,v'_1)+(v_1-v'_1)\mathcal{E}_{v_2\in[\underline{v},v'_1]}[X_1(v'_1,v_2)]+\mathcal{E}_{v_2\in(v'_1,v_1)}[(v_1-v_2)X_1(v'_1,v_2)], \tag{11}$$

and for all $v_1 \leq v'_1$:

$$u_1(v_1,v_1) \geq u_1(v'_1,v'_1)+(v_1-v'_1)\mathcal{E}_{v_2\in[\underline{v},v_1]}[X_1(v'_1,v_2)]+\mathcal{E}_{v_2\in(v_1,v'_1)}[(v_2-v'_1)X_1(v'_1,v_2)]. \tag{12}$$

Then, for any two points v_1 and $v'_1 \leq v_1$, we must have

$$u_1(v_1,v_1) \geq u_1(v'_1,v'_1)+(v_1-v'_1)\mathcal{E}_{v_2\in[\underline{v},v'_1]}[X_1(v'_1,v_2)]+\mathcal{E}_{v_2\in(v'_1,v_1)}[(v_1-v_2)X_1(v'_1,v_2)], \tag{13}$$

$$u_1(v'_1,v'_1) \geq u_1(v_1,v_1)+(v'_1-v_1)\mathcal{E}_{v_2\in[\underline{v},v'_1]}[X_1(v_1,v_2)]+\mathcal{E}_{v_2\in(v'_1,v_1)}[(v_2-v_1)X_1(v_1,v_2)]. \tag{14}$$

Combining these equations, to induce truth telling, the variation in equilibrium utility between these two points must satisfy

$$(v_1-v'_1)\mathcal{E}_{v_2\in[\underline{v},v'_1]}[X_1(v'_1,v_2)]+\mathcal{E}_{v_2\in(v'_1,v_1)}[(v_1-v_2)X_1(v'_1,v_2)] \leq u_1(v_1,v_1)-u_1(v'_1,v'_1)$$
$$\leq (v_1-v'_1)\mathcal{E}_{v_2\in[\underline{v},v'_1]}[X_1(v_1,v_2)]+\mathcal{E}_{v_2\in(v'_1,v_1)}[(v_1-v_2)X_1(v_1,v_2)],$$

which can be rewritten as (by manipulating the r.h.s.):

$$(v_1-v'_1)\mathcal{E}_{v_2\in[\underline{v},v'_1]}[X_1(v'_1,v_2)]+\mathcal{E}_{v_2\in(v'_1,v_1)}[(v_1-v_2)X_1(v'_1,v_2)] \leq u_1(v_1,v_1)-u_1(v'_1,v'_1)$$
$$\leq (v_1-v'_1)\mathcal{E}_{v_2\in[\underline{v},v_1]}[X_1(v_1,v_2)]+\mathcal{E}_{v_2\in(v'_1,v_1)}[(v'_1-v_2)X_1(v_1,v_2)], \tag{15}$$

and this requires in particular

$$(v_1-v_1')\mathcal{E}_{v_2\in[\underline{v},v_1']}[X_1(v_1',v_2)]+\mathcal{E}_{v_2\in(v_1',v_1)}[(v_1-v_2)X_1(v_1',v_2)]\le (v_1-v_1')\mathcal{E}_{v_2\in[\underline{v},v_1]}[X_1(v_1,v_2)]+\mathcal{E}_{v_2\in(v_1',v_1)}[(v_1'-v_2)X_1(v_1,v_2)] \quad \forall v_1'\le v_1. \tag{16}$$

A necessary condition for the new monotonicity condition (16) to hold is

$$\mathcal{E}_{v_2\in[\underline{v},v_1']}[X_1(v_1',v_2)]\le \mathcal{E}_{v_2\in[\underline{v},v_1]}[X_1(v_1,v_2)] \quad \forall v_1'\le v_1. \tag{17}$$

That is, $\mathcal{E}_{v_2\in[\underline{v},v_1]}[X_1(v_1,v_2)]$ must be non-decreasing, hence it is differentiable almost everywhere. The envelop theorem applied to the maximization of $u_1(v_1,v_1')$ with respect to v_1' yields

$$\frac{d}{dv_1}u_1(v_1)=\int_{\underline{v}}^{v_1} X_1(v_1,v_2)\,dv_2=\mathcal{E}_{v_2\in[\underline{v},v_1]}[X_1(v_1,v_2)]. \tag{18}$$

The two conditions (18) and (16) together imply (11) and (12) as long as $v_1'\to v_1$. That is, they ensure that telling the truth is a local optimum, and they are necessary for a global optimum. Given this restriction, global optimality is satisfied if (15) holds, which can be written as $(\widehat{IC})$ by using (18). The proof is similar for agent 2.

Last, at equilibrium, the expected transfer paid by agent 1 is

$$E_{v_2}t_1(v_1,v_2)=v_1\mathcal{E}_{v_2\in[\underline{v},v_1]}[X_1(v_1,v_2)]+\mathcal{E}_{v_2\in]v_1,\overline{v}]}[v_2X_1(v_1,v_2)]-\alpha(v_0)E_{v_2}X_0(v_1,v_2)-u_1(v_1).$$

Using (18), the equilibrium utility is

$$u_1(v_1)=\int_{\underline{v}}^{v_1}\mathcal{E}_{v_2\in[\underline{v},s]}[X_1(s,v_2)]\,ds+u_1(\underline{v}),$$

where $u_1(\underline{v})=-\alpha(v_0)$. We use a similar method to rewrite the expected transfer paid by agent 2. Replacing these expressions in the expected revenue of the seller and integrating by parts, we obtain the objective function of the optimization program. The remaining constraints are $(\widehat{IC})$ and the feasibility constraints (F). □

Proof of Proposition 6. Neglecting $(\widehat{\text{IC}})$, the mechanism that maximizes the unconstrained relaxed problem is $\mathcal{M}^{unc}$

$$\begin{aligned}
&X_1(v)=1 \quad \text{if } v_1\le v_2 \text{ and } v_2>r_2^F,\\
&X_2(v)=1 \quad \text{if } v_2\le v_1 \text{ and } v_1\ge r_1^F,\\
&X_0(v)=1 \quad \text{otherwise.}
\end{aligned}$$

This is the case because when $v_i<v_j$, $\pi_i^*(v_i)<\pi_i^F(v_i)$ and it is therefore best to set $X_j(v)=1$ and $X_i(v)=0$. This mechanism is deterministic and not resale-proof. However this mechanism violates $(\widehat{\text{IC}})$, as it trivially violates the necessary condition (17). This can be seen by direct inspection of the condition. A necessary condition to restore $(\widehat{\text{IC}})$ is to allocate the good to the agent with the highest valuation sometimes with a positive probability.

For future reference, we want to note that $\mathcal{A}^{RI}$ satisfies $(\widehat{\text{IC}})$. To see this, we check $(\widehat{\text{IC}})$ for all pairs (v_i',v_i). Consider agent 1 for instance. For all $v_1'<v_1\le r_1^*$, the good is not allocated to either v_1 or v_1', we have $X_1(v_1,v_2)=X_1(v_1',v_2)=0$ and $(\widehat{\text{IC}})$ is satisfied. For all $v_1'\le r_1^*$ and $v_1>r_1^*$, $(\widehat{\text{IC}})$ writes as (the calculation requires one integration by parts in the right hand side of the constraint)

$$0\le\int_{r_1^*}^{v_1}F(s)\,ds\le\int_{v_1'}^{v_1}F(s)\,ds,$$

which is true. For all $r_1^*<v_1'<v_1$, $(\widehat{\text{IC}})$ writes as (again, the calculation requires one integration by parts)

$$\int_{v_1'}^{v_1}F(s)\,ds\le\int_{v_1'}^{v_1}F(s)\,ds\le\int_{v_1'}^{v_1}F(s)\,ds,$$

which is also true.

Consider values $v_i>v_j$. To restore $(\widehat{\text{IC}})$, we need to allow the seller to distort the allocation in $\mathcal{M}^{unc}$. From a general perspective, the seller can do three things: keep the good, allocate to i and allocate to j. Therefore, let us assume that the seller keeps the good with probability $p_0(v)$, allocates the good to i with probability $p_i(v)$, and to j with probability $1-p_0(v)-p_i(v)$.

We first construct allocations that provide the seller with virtual surplus that are at least positive (at all points), and we denote this class by $\mathbf{A}^+$. If $v_i>r_i^*$, it is best to allocate the good to either agent (both virtual surplus are positive) rather than keeping it. Therefore, mechanisms in $\mathbf{A}^+$ are such that $p_0(v)=0$, $p_i(v)\ge0$ and $p_j(v)\ge0$. If $v_i<r_i^F$, it is best to not allocate the good at all (both virtual surplus are negative) and mechanisms in $\mathbf{A}^+$ are such that $p_0(v)=1$. When $v_i\in(r_i^F,r_i^*)$, it is best to not allocate the good to i and mechanisms in $\mathbf{A}^+$ are such that $p_i(v)=0$, $p_0(v)\ge0$ and $p_j(v)\ge0$. By construction, this type of mechanism can be interpreted as a randomization between $\mathcal{M}^{unc}$ and $\mathcal{A}^{RI}$ where $q(v)$ is the probability of applying mechanism $\mathcal{A}^{RI}$. To see this, if $v_i>r_i^*$, with probability $p_i(v)=q(v)$, i obtains the good as prescribed by $\mathcal{A}^{RI}$, and with probability $1-q(v)$, j obtains the good as prescribed by $\mathcal{M}^{unc}$. When $v_i<r_i^F$, both mechanisms prescribe to not allocate the good. When $v_i\in(r_i^F,r_i^*)$, with probability $q(v)$, the good is not allocated as prescribed by $\mathcal{A}^{RI}$, and with probability $1-q(v)$, j obtains the good as prescribed by $\mathcal{M}^{unc}$. Last, there exist mechanisms in $\mathbf{A}^+$ that satisfy $(\widehat{\text{IC}})$: as noted before $\mathcal{A}^{RI}$

satisfies the constraint and belong to that class (obtained for $q(v)=1$ for all v) . This mechanism is the unique resale-proof mechanism in the class we consider.

Assume that the seller selects a mechanism in $\mathbf{A}^+$. Can she do better by deviating to a different mechanism? The only mechanisms not contained in $\mathbf{A}^+$ are mechanisms that provide the seller with a negative surplus sometimes. We now show that the optimal mechanism needs to be in $\mathbf{A}^+$. For convenience, let us rewrite ($\widehat{\mathrm{IC}}$) (the r.h.s. has been rearranged)

$$(v_1-v_1')\mathcal{E}_{v_2\in[\underline{v},v_1']}[X_1(v_1',v_2)]+\mathcal{E}_{v_2\in(v_1',v_1)}[(v_1-v_2)X_1(v_1',v_2)] \le \int_{v_1'}^{v_1} \mathcal{E}_{v_2\in[\underline{v},s]}X_1(s,v_2)\,ds$$
$$+ \le (v_1-v_1')\mathcal{E}_{v_2\in[\underline{v},v_1']}[X_1(v_1,v_2)]+\mathcal{E}_{v_2\in(v_1',v_1)}[(v_1-v_2)X_1(v_1,v_2)].$$

(a) Consider a mechanism in $\mathbf{A}^+$ that satisfy ($\widehat{\mathrm{IC}}$). Suppose the seller decides to allocate the good with positive probability to agent 1 at point (v_1,v_2) such that $v_1>v_2$ and $v_1<r_1^*$. Such transaction yields negative surplus. This distortion of the original mechanism has the following effects on incentive compatibility (by inspection of ($\widehat{\mathrm{IC}}$)). First, for all $v_1'<v_1$, ($\widehat{\mathrm{IC}}$) is still satisfied (although the rent of the agent increases). Second, for all $v_1'>v_1$, ($\widehat{\mathrm{IC}}$) may require to increase also the probability of allocating the item to v_1' (replacing v_1 by v_1' in ($\widehat{\mathrm{IC}}$), the l.h.s. increases and the r.h.s. may need to increase as well). This implies allocating more often to v_1', that is generating a negative surplus, or decreasing the probability of allocating to agent 2. However, remember that selling to agent 2 in that region provides the highest surplus (according to the unconstrained mechanism $\mathcal{M}^{unc}$). Overall, for any mechanism in $\mathbf{A}^+$ that satisfies ($\widehat{\mathrm{IC}}$), there does not exist a mechanism in its complement $\overline{\mathbf{A}^+}$ that does strictly better: distorting the original mechanism to accommodate inefficient trades is detrimental.

(b) Consider a mechanism that satisfies ($\widehat{\mathrm{IC}}$) but allocates the good with negative surplus sometimes. Suppose agent 1 obtains the good with positive probability at point (v_1,v_2) such that $v_1>v_2$ and $v_1<r_1^*$. Assume that this probability is reduced. First, for all $v_1'<v_1$, ($\widehat{\mathrm{IC}}$) may require to decrease also the probability of allocating the item to v_1'. This implies allocating less often to v_1' that generates a negative surplus. Second, for all $v_1'>v_1$, ($\widehat{\mathrm{IC}}$), is still satisfied (although the rent of the agent decreases). Overall, it is possible to find an allocation in $\mathbf{A}^+$ that also satisfies ($\widehat{\mathrm{IC}}$) and dominates the original allocation.

This proves that the optimal mechanism is in $\mathbf{A}^+$. We now need to show that the particular case $\mathcal{A}^{RI}$ is dominated in that class. Consider for instance mechanisms such that $q(v)=q<1$ for all v. Consider v_1 and v_1' such that both mechanisms allocate the good to both types (similar algebra can be performed for the other cases and is omitted), then ($\widehat{\mathrm{IC}}$) can be rewritten as

$$(v_1-v_1')qF(v_1')+v_1(1-q)[F(v_1)-F(v_1')]-\int_{v_1'}^{v_1} v_2(1-q)\,dF(v_2)\le q\int_{v_1'}^{v_1}F(s)\,ds$$
$$\le (v_1-v_1')qF(v_1)+v_1'q[F(v_1)-F(v_1')]-\int_{v_1'}^{v_1} v_2q\,dF(v_2).$$

Simple calculations show that this is satisfied for all $q\ge 1/2$. Given $\pi_i^F(v_i)\ge \pi_i^*(v_i)$, such mechanisms generate a higher revenue than $\mathcal{A}^{RI}$. This implies that the optimal mechanism is not resale-proof.

We have shown that the solution to the problem of the seller is a mechanism that randomizes between $\mathcal{M}^{unc}$ and $\mathcal{A}^{RI}$. The probability $q(v)$ must be chosen so that ($\widehat{\mathrm{IC}}$) is satisfied (note that a necessary condition is for $q(v)$ to satisfy (17)). □

Appendix B

In this appendix, we consider an allocation procedure based on a second price sealed bid auction to implement the optimal mechanism in Proposition 1. Assume first that the seller allocates the good through a second-price sealed bid auction with reserve prices r_1 and r_2 faced by bidders 1 and 2, respectively.

Lemma 4. *In any second price sealed bid auction with reserve prices, the optimal bidding strategies are $b_i(v_i)=v_i$ for all $v_i\ge r_i$. Besides, there exists $\hat{v}_1(r_1,r_2)<r_1$ increasing in r_1 and decreasing in r_2 such that $b_1(v_1)=r_1$ for all $v_1\in(\hat{v}_1(r_1,r_2),r_1)$.*

Proof. Suppose the seller allocates the good through a second-price sealed bid auction with reserve prices. The aim is to determine the bidding strategies of bidders, and the optimal reserve prices. Agent 1 anticipates that agent 2 bids $b_2(v_2)$, where $b_2(\cdot)$ is increasing in v_2. Suppose the reserve prices are r_1 and r_2 for agents 1 and 2, respectively. If agent 1 bids b_1 and gets the good, his surplus is $v_1-\max\{r_1,b_2(v_2)\}$. If he does not get it, either 2 acquires it in which case the surplus of agent 1 is 0, or the seller keeps it in which case agent 1's surplus is $-\alpha(v_0)$. Agent 1 wins if $b_2(v_2)<b_1$ provided $b_1>r_1$. The seller keeps the good if $b_2(v_2)<r_2$ and $b_1<r_1$. Let $u_1(v_1,b_1)$ be the expected utility of agent 1 when his valuation is v_1 and he bids b_1, we have

$$u_1(v_1,b_1)=\begin{cases} v_1F(b_2^{-1}(b_1))-\int_{b_2^{-1}(r_1)}^{b_2^{-1}(b_1)} b_2(s)\,dF(s)-r_1F(b_2^{-1}(r_1)) & \text{if } b_1>r_1,\\ -\alpha(v_0)F(b_2^{-1}(r_2)) & \text{otherwise.}\end{cases}$$

Consider $b_1>r_1$. Agent 1 chooses b_1 such that $\partial/\partial b_1 u_1(v_1,b_1)=0$. The function is concave with a maximum in v_1. Then, the optimal bidding strategy is $b_1=v_1$ for all $v_1>r_1$. If $v_1<r_1$ and agent 1 bids $b_1<r_1$, then his utility is $-\alpha(v_0)F(b_2^{-1}(r_2))$.

Conditional on bidding above the reserve price, his best strategy is to bit $b_1 = r_1$. There exists $\hat{v}_1 < r_1$ such that

$$\hat{v}_1 F(b_2^{-1}(r_1)) - r_1 F(b_2^{-1}(r_1)) = -\alpha(v_0) F(b_2^{-1}(r_2)).$$

For all $v_1 \in [\hat{v}_1, r_1]$, $b_1 = r_1$ and for all $v_1 < \hat{v}_1$, $b_1 < r_1$ and the bid is irrelevant. The argument is similar for agent 2:

$$u_2(v_2, b_2) = \begin{cases} v_2 F(b_1^{-1}(b_2)) - \int_{b_1^{-1}(r_2)}^{b_1^{-1}(b_2)} b_1(s)\, dF(s) - r_2 F(b_1^{-1}(r_2)) & \text{if } b_2 > r_2, \\ 0 & \text{otherwise,} \end{cases}$$

and the optimal bid is $b_2 = v_2$. Given there is no externality, the bid of agent 2 is irrelevant if $v_2 < r_2$. In equilibrium, we have

$$\hat{v}_1(r_1, r_2) F(r_1) - r_1 F(r_1) = -\alpha(v_0) F(r_2).$$

differentiating this expression with respect to r_1 and r_2, $\hat{v}_1(r_1, r_2)$ is increasing with respect to r_1 and decreasing with respect to r_2. $\square$

The intuition of this result is as follows. First, bidder i wins at the correct time against bidder j if he bids his valuation. This is the case because there is no externality between bidders. Second, bidder 1 is willing to pay to avoid the seller from keeping the good. As long as bidder 1's valuation is above his reserve price, bidding his valuation guarantees the seller does not keep the good. However, if his valuation is below his reserve price, bidding his valuation is not enough and he needs to rely on bidder 2's bid. Therefore, it may be beneficial for bidder 1 to increase his bid up to r_1 at the risk of obtaining the good at too high a price: this strategy acts as an insurance against the externality. Overall, if bidder 1's valuation is in the interval $(\hat{v}_1(r_1, r_2), r_1)$, he prefers to bid r_1. At $\hat{v}_1(r_1, r_2)$, the agent is indifferent between bidding (and making an expected loss because the good is too expensive compared to his valuation) and not bidding (and suffering the externality with a positive probability). Last, consider an agent with valuation $\hat{v}_1(r_1, r_2)$. When the reserve price r_1 increases, the option of bidding r_1 becomes less beneficial. When r_2 increases, bidder 2 is less likely to obtain the good and the seller is more likely to keep the good. Then, the option of bidding r_1 becomes more attractive. Overall, the cutoff point below which it is not optimal to bid increases in r_1 and decreases in r_2.

Proposition 7. *The seller can implement the optimal mechanism with a modified second-price sealed bid auction where*:

(i) *Agent 1 first pays an entry fee $c_1 = \alpha(v_0)$. If he participates, he faces the reserve price r_1^* and gets the good if $b_1 > r_1^*$ and $b_1 > h^{-1}(b_2)$. If he wins, he pays* $\max\{r_1^*, h^{-1}(b_2)\}$. *If he bids below r_1 (or does not bid), he receives a subsidy $s_1 = \alpha(v_0) F(r_2^*)$.*
(ii) *Agent2 faces the reserve price r_2^* and gets the good if $b_2 > r_2^*$ and $b_2 > h(b_1)$. If he wins, he pays* $\max\{r_2^*, h(b_1)\}$.

The optimal bidding strategies are $b_1 = v_1$ if $v_1 > r_1$ and $b_2 = v_2$ if $v_2 > r_2$.

Proof. Let us consider a modified second-price auction with the following features. Agent 1 anticipates that agent 2 bids $b_2(v_2)$, where $b_2(\cdot)$ is increasing in v_2. If both agents bid above the reserve prices, agent 1 gets the good if $b_1 \geq k \circ b_2(v_2)$, in which case he pays $k \circ b_2(v_2)$. If agent 1 bids b_1 and gets the good, his surplus is $v_1 - \max\{r_1, k \circ b_2(v_2)\}$. If he does not get it, either 2 acquires it in which case the surplus of agent 1 is 0, or the seller keeps it in which case agent 1's surplus is $-\alpha(v_0)$. Let $u_1(v_1, b_1)$ be the expected utility of agent 1 when his valuation is v_1 and he bids b_1, we have

$$u_1(v_1, b_1) = \begin{cases} v_1 F(b_2^{-1} \circ k^{-1}(b_1)) - \int_{b_2^{-1} \circ k^{-1}(r_1)}^{b_2^{-1} \circ k^{-1}(b_1)} k \circ b_2(s) dF(s) - r_1 F(b_2^{-1} \circ k^{-1}(r_1)) & \text{if } b_1 > r_1, \\ -\alpha(v_0) F(b_2^{-1}(r_2)) & \text{otherwise.} \end{cases}$$

Agent 1 chooses b_1 such that $\partial/\partial b_1 u_1(v_1, b_1) = 0$. For all $b_1 > r_1$, the optimal bidding strategy is $b_1 = v_1$, provided $v_1 > r_1$. Note also that the equilibrium utility is increasing in v_1 and at $v_1 = r_1$, it is 0. If $v_1 < r_1$, agent 1 cannot do better than bid exactly r_1. For all $b_1 < r_1$, agent 1's utility is $-\alpha(v_0) F(b_2^{-1}(r_2))$. There exists $\hat{v}_1 < r_1$ such that

$$\hat{v}_1 F(b_2^{-1} \circ k^{-1}(r_1)) - r_1 F(b_2^{-1} \circ k^{-1}(r_1)) = -\alpha(v_0) F(b_2^{-1}(r_2)).$$

For all $v_1 \in [\hat{v}_1, r_1]$, $b_1 = r_1$ and for all $v_1 < \hat{v}_1$, $b_1 < r_1$ (and the bid is irrelevant).

The argument is similar for agent 2:

$$u_2(v_2, b_2) = \begin{cases} v_2 F(b_1^{-1} \circ k(b_2)) - \int_{b_1^{-1} \circ k(r_2)}^{b_1^{-1} \circ k(b_2)} k^{-1} \circ b_1(s)\, dF(s) - r_2 F(b_1^{-1} \circ k(r_2)) & \text{if } b_2 > r_2, \\ 0 & \text{otherwise,} \end{cases}$$

and the optimal bid is $b_2 = v_2$. Given there is no externality, the bid of agent 2 is irrelevant if $v_2 < r_2$. Overall, in equilibrium we have

$$u_1(v_1) = \begin{cases} v_1 F(k^{-1}(v_1)) - \int_{k^{-1}(r_1)}^{k^{-1}(v_1)} k(s)\, dF(s) - r_1 F(k^{-1}(r_1)) & \text{if } v_1 > r_1, \\ v_1 F(k^{-1}(r_1)) - r_1 F(k^{-1}(r_1)) & \text{if } v_1 \in [\hat{v}_1, r_1], \\ -\alpha(v_0) F(r_2) & v_1 < \hat{v}_1, \end{cases}$$

$$u_2(v_2)=\begin{cases} v_2F(k(v_2))-\int_{r_1}^{k(v_2)}k^{-1}(s)\,dF(s)-k^{-1}(r_1)[F(r_1)-F(\hat{v}_1)]-r_2F(\hat{v}_1) & \text{if } v_2>r_1,\\ v_2F(\hat{v}_1)-r_2F(\hat{v}_1) & \text{if } v_2\in[r_2,r_1],\\ 0 & \text{otherwise}.\end{cases}$$

In the optimal mechanism agent 1 acquires the good if $v_1>h^{-1}(v_2)$ provided that both valuations are above the respective reserve prices. Therefore, we need $k=h^{-1}$, $r_1=r_1^*$ and $r_2=r_2^*$. Also, we need to make sure that agents below r_1 do not obtain the good. All agents in $[\hat{v}_1,r_1]$ bid r_1 and obtain the good with positive probability. They still get a negative payoff but it is greater than $-\alpha(v_0)F(r_2)$. Suppose we set a subsidy of $\alpha(v_0)F(r_2)$ when $b_1<r_1$. Then, they get 0 by not bidding. With this subsidy, $\hat{v}_1=r_1$. Overall, the equilibrium utilities in the auction are

$$u_1(v_1)=\begin{cases} v_1F(h(v_1))-\int_{r_2^*}^{h(v_1)}h^{-1}(s)\,dF(s)-r_1^*F(r_2^*) & \text{if } v_1>r_1^*,\\ 0 & \text{otherwise},\end{cases}$$

$$u_2(v_2)=\begin{cases} v_2F(h^{-1}(v_2))-\int_{r_1^*}^{h^{-1}(v_2)}h(s)\,dF(s)-r_2^*F(r_1^*) & \text{if } v_2>r_1^*,\\ v_2F(r_1^*)-r_2^*F(r_1^*) & \text{if } v_2\in[r_2^*,r_1^*],\\ 0 & \text{otherwise}.\end{cases}$$

We need to check whether the appropriate transfers are implemented. Consider first agent 1. The expected transfer of agent 1 if his valuation is $v_1<r_1^*$ is $-\alpha v_0F(r_2^*)$. In the optimal auction he would pay $E_{v_2}t_1^*(v)=-\alpha(v_0)F(r_2^*)+\alpha(v_0)$. An agent with valuation $v_1>r_1^*$ pays

$$\int_{r_2^*}^{h(v_1)}h^{-1}(s)\,dF(s)+r_1F(r_2^*)=v_1F(h(v_1))-\int_{r_2^*}^{h(v_1)}\frac{dh^{-1}(s)}{ds}F(s)\,ds.$$

In the optimal auction, he would pay

$$E_{v_2}t_1^*(v)=v_1F(h(v_1))-\int_{r_1^*}^{v_1}F(h(s))\,ds+\alpha v_0.$$

Let $u=h(s)$, the transfer in the optimal auction is simply:

$$v_1F(h(v_1))-\int_{r_2^*}^{h(v_1)}\frac{dh^{-1}(u)}{du}F(u)\,du+\alpha v_0.$$

To implement the optimal mechanism, the seller must set an entry fee equal to $c_1=\alpha v_0$.

Consider now agent 2. His expected transfer if his valuation is $v_2<r_2^*$ is 0 in both the auction and the optimal mechanism. When $v_2\in[r_2^*,r_1^*]$, he pays $r_2^*F(r_1^*)$ in the auction. His expected payment in the optimal auction is

$$E_{v_1}t_2^*(v)=v_2F(r_1)-\int_{r_2^*}^{v_2}F(r_1^*)\,ds=r_2^*F(r_1^*).$$

When $v_2>r_1^*$, agent 2 pays

$$\int_{r_1^*}^{h^{-1}(v_2)}h(s)\,dF(s)+r_2^*F(r_1^*)=v_2F(h^{-1}(v_2))-\int_{r_1^*}^{h^{-1}(v_1)}\frac{dh(s)}{ds}F(s)\,ds$$

and

$$E_{v_1}t_2^*(v)=v_2F(h^{-1}(v_2))-\int_{r_2^*}^{v_2}F(h^{-1}(s))\,ds.$$

Again, let $u=ds$, the optimal transfer in the optimal mechanism is simply

$$E_{v_1}t_2^*(v)=v_2F(h^{-1}(v_2))-\int_{r_1^*}^{h^{-1}(v_1)}\frac{dh(s)}{ds}F(s)\,ds.\quad\square$$

Given it is optimal to have asymmetric reserve prices in the optimal auction (see Proposition 1), it is necessary to use those in the modified Vickrey auction as well. Then $r_1=r_1^*$ and $r_2=r_2^*$. Besides, conditional on both valuations (and in that case both bids) being above their respective reserve prices, the seller wants to favor bidder 2. Therefore, she must compare functions of bids rather than bids themselves.[36]

[36] The case of Valencia C.F. points to the fact the team resorted to an asymmetric allocation rule. Still, it is not possible to determine whether Real Madrid C.F. did not win because its bid was below the reserve price ($b_1<r_1^*$) or because it did not compare favorably to the bid of S.S. Lazio ($b_1<h^{-1}(b_2)$).

Interestingly, it is also necessary to subsidize bidder 1 when he does not bid enough: in the optimal mechanism, an agent with valuation below r_1 never enjoys the good and suffers the externality any time the seller keeps it (which happens when bidder 2's valuation is below r_2). An auction that implements the optimal mechanism needs to reproduce that feature. Consider now a modified second-price sealed bid auction where bidder 1 wins if $b_1 > r_1^*$ and $b_1 > h^{-1}(b_2)$ and bidder 2 wins if $b_2 > r_2^*$ and $b_2 > h(b_1)$. If bidder 1's type is exactly r_1^*, then it is optimal for him to bid exactly that value. He wins and pays r_1^* if and only if his opponent bids below $h(r_1^*)$ and he loses otherwise (against bidder 2 who does not exert any externality on him). His expected payoff is therefore 0 and the seller will never keep the good. If he does not bid (or bids below r_1^*), then the seller might keep the good. This occurs if bidder 2's valuation is below r_2^*. Then bidder 1's expected payoff is $-\alpha(v_0)F(r_2^*)$. Overall, an agent with valuation r_1^* is strictly better-off by bidding r_1^*. Therefore, there exists an interval $(\hat{v},r_1^*)$ such that a bidder with a valuation in that interval bids r_1^* and might obtain the good. To make sure this does not happen, it is necessary to subsidize that agent and make him indifferent between bidding r_1^* and not bidding at all. Last, given the presence of externalities between the seller and bidder 1, it is possible to extract additional payments not reflected in the bidding strategy. In the optimal mechanism, the utility of an agent with type below r_1^* is equal to the outside option: it is as if he suffers the externality for sure. However, in the modified sealed bid auction, he suffers the externality only with some probability. Then, the seller needs to resort to extra fees to capture the difference, or said differently, to make sure the utility levels are shifted downwards. Naturally, such fees are not imposed on bidder 2. Overall, bidder 1 needs to pay an entry fee (that is only partially recouped when he bids below r_1^*). Note that this feature is also present in auctions implementing optimal mechanisms in the presence of externalities between bidders.

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