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Countervailing incentives in allocation mechanisms with type-dependent externalities☆

Isabelle Brocas *

University of Southern California, USA
CEPR, United Kingdom

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ABSTRACT

I study an allocation mechanism of a single item in the presence of type-dependent externalities between bidders. The type-dependency introduces countervailing incentives and the allocation sometimes requires that types in an interior subset obtain their reservation utility. Furthermore, truth-telling requires the ex-ante allocation to satisfy a non-trivial monotonicity condition. I show that this problem is technically different from the one analyzed in related single agent settings. I provide a procedure to identify the main properties of the ex-post allocation. Typically, the solution does not entail a single reserve price. More specifically, each agent faces an allocation rule contingent on whether his and his rival's types fall below, in or above the (endogenously determined) subset of types that obtain their reservation utility.

1. Introduction

Consider two firms in a market competing for the acquisition of a license. There are three possible outcomes for each firm. The license may not be allocated at all, in which case both firms get the status quo payoff; the firm wins the license and its payoff increases; or the rival wins and the payoff of the firm decreases. Overall, the winner induces a *negative externality* on the loser and the designer of the allocation mechanism should reflect this in her pricing strategy. The effect of negative externalities on prices has been studied in the literature starting with Katz and Shapiro (1986) and Kamien et al. (1992). As shown in these early studies, the asymmetric ex-post interaction between firms allows the seller to extract some payments even from the firms that do not obtain the license. The analysis has been generalized by Jehiel et al. (1996, 1999) with the characterization of the optimal allocation mechanism when firms have private information about their valuation for the good, and extended by various authors. In most of this literature, externalities are taken to be unrelated to the agent's valuation of the good.[1] The optimal mechanism exhibits the same qualitative properties as the standard optimal auction without externalities.[2] The only difference is that the seller can extract payments from non acquirers (which can be implemented via an entry fee), and this extra payment as well as the reserve price increase in the externality.

Yet, it is difficult to think of realistic games and markets in which valuations and externalities are unrelated. To see this, consider again the licensing example. The valuation for the license and the externality suffered when a rival obtains it depends on the intrinsic ability of the firm to both exploit the innovation and sustain competition from a rival licensee. This suggests that valuation for the good, suffered externality and imposed externality are linked through underlying variables, and that the specific structure of the industry will determine the sign and amount of the correlation. Carrillo (1998) was the first to analyze that possibility and to characterize the optimal contract with multiple agents and valuation-dependent externalities.[3] In that case, the reservation utility of the

☆ This paper builds on comments made in Brocas (1998) and exploits one case analyzed in Brocas (2001–2009). I am grateful to Juan Carrillo, Harrison Cheng, Hugo Hopenhayn, John Riley, Guofu Tan, seminar participants at the University of Southern California, the University of California Los Angeles, the University of Southampton, the University of Edinburgh and the South West Economic Theory Conference, the Coeditor in charge of the manuscript and two anonymous referees for useful comments on earlier versions.

* Correspondence to: Department of Economics, University of Southern California, 3620 S. Vermont Ave. Los Angeles, CA 90089-0253, USA.

E-mail address: brocas@usc.edu.

1 In Jehiel et al. (1996), the size of the externality suffered by an agent is unknown to him but depends only on the identity of the winner and not on his valuation. In Jehiel et al. (1999), the externality suffered by an agent is private information and depends on the identity of the winner but it is not correlated to his valuation. Other mechanism design problems consider identity-dependent externalities (e.g. Aseff and Chade, 2006 for the case of multi-unit auctions). Most of these papers will be discussed in further detail.

2 See Myerson (1981) for the seminal paper on optimal auctions and Engelbrecht-Wiggans (1980), McAfee and McMillan (1987) and Klemperer (1999) for surveys.

3 Valuation-dependent externalities have also been studied by Jehiel and Moldovanu (2000). However, the authors restrict to particular procedures and the analysis is not relevant to the optimal allocation problem.

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agent becomes valuation-dependent (the agent suffers the externality if he does not show up to bid and the rival is allocated the good). This extra feature gives rise to countervailing incentives. At equilibrium, the utility of a participating firm may bind at the bottom, at the top or for interior valuations on the reservation utility. Carrillo (1998) restricts attention to functional forms that allow only for the two first situations which correspond to cases in which the reservation utility is respectively flat or steep.[4] Closely related, Figueroa and Skreta (2009) studies the role of optimal threats when several outside options may coexist. The authors consider three cases, two of them being equivalent to two cases already analyzed in Carrillo (1998) (the reservation utility is flat always, or steep always).[5] The novelty arises in one case, when the reservation utility can be either flat or steep. The authors show that the seller must randomize between the two reservation utilities, and the equilibrium utility may bind for interior types. They provide the solution in a numerical example in which only one bidder has private information, reducing the problem to a single agent mechanism design problem.

The objective of this paper is to study the problem when agents have a single but type-dependent outside option, *both* remain privately informed, and the equilibrium utility binds for interior types. I therefore consider the case where the reservation utility is weakly increasing (a case in-between flat and steep) in the valuation.[6] This problem is interesting because extreme cases obtain rarely in concrete examples. Also, mechanisms in which the equilibrium utility binds for interior types usually exhibit different properties from mechanisms in which it binds at a boundary point of the support. The study of extreme cases is therefore not without loss of generality and it is important to address properly the more likely scenarii. The problem turns out to be challenging to solve and I characterize the main properties of the optimal ex-post allocation mechanism. The mechanism must be such that the agent receives no rent for valuations at which the equilibrium utility binds, hereafter called the 'set of binding types'. Also, the mechanism requires some form of monotonicity of the ex-ante allocation to make sure truth-telling is a global maximum. These two requirements act as constraints on the optimization program of the seller.

I start by studying an unconstrained optimization program and show that its solution violates both constraints. This is the case because the allocation to the right of each bound of the set of binding types differs from the allocation to the left. There are two reasons for this. First, there is a tension between the rents that must be given below the set of binding types and above it. Indeed, giving one extra unit of rent to a given type requires to also increase the rents of other types, which are in different proportions if types lie below or above the set of binding types. Second, there is also a tension between the constraints and the interest of the seller on the set of binding types. Indeed, those types must receive no rent but the efficient solution requires to leave rents under asymmetric information.

I then develop a procedure to identify a set of properties of the optimal ex-post allocation. The procedure consists in fixing a pair of sets of binding types and focusing on mechanisms that satisfy the constraints while maximizing revenue. By varying the sets of binding types, the optimal mechanism is the one yielding the highest revenue overall. Even though the procedure does not allow to fully characterize the optimal mechanism, it permits to identify two novel properties. Contrary to standard allocation problems, the solution does not entail a single reserve price. Rather, each agent faces an allocation rule contingent on whether he and his rival's types fall below, in or above the set of binding types. This is the case because of the difference between the rents that must be given below the set of binding types and above it. The value of allocating the good to an agent with a valuation falling below, in, or above the set of binding types is therefore different for the seller, calling for different rules. Moreover, at equilibrium, the agent with the highest type does not necessarily obtain the good and the seller will resort to a stochastic mechanism for some pairs of types. This occurs sometimes because, even though allocating the good to one agent generates a higher surplus, this allocation conflicts with the constraints. It is therefore optimal to distort the unconstrained allocation. These results are different from what has been obtained previously in the literature on auctions with externalities. This suggests that a simple modification of the externality model may impact dramatically the predictions. Section 2 presents the model and Section 3 solves for the optimal mechanism. All proofs are in the Appendix.

2. The model

An indivisible good is offered for sale among two risk-neutral potential buyers 1 and 2, indexed by i and j. Buyer i (he) derives utility v_i when he gets the good. We will call v_i, his "willingness to pay", "type" or "valuation" and $v = (v_i, v_j)$ the vector of valuations of both agents. Each v_i is drawn independently from a common knowledge distribution defined on the interval $[\underline{v}, \overline{v}]$, with $0 < \underline{v} < \overline{v}$.

Assumption 1. Valuations v_i are drawn from a uniform distribution.

This assumption is made to simplify the main argument. Let $\Delta = \overline{v} - \underline{v}$. The valuation for the good of the seller (she) is zero. Bidder i suffers an externality $-\alpha_i(v)$ when bidder $j \neq i$ gets the good. This externality is always negative and is a function of both the valuation of the agent who gets the good (v_j) and that of the agent who suffers from not getting it (v_i). In order to keep the analysis as tractable as possible, we shall restrict to the following linear form:

Assumption 2. $\alpha_i(v) = \alpha_a v_i + \alpha_b v_j + \gamma$ where α_a, α_b and γ are such that $\alpha_i(v) > 0 \; \forall \, v_i, v_j$.

Under asymmetric information, the reservation utility of each agent is given by the outcome of the auction if he does not show up, so it is mechanism dependent. In the presence of negative externalities, each agent wants not only to acquire the good, but also to avoid the externality that results when the rival gets it. Then, he is prone to pay and enter the auction if participating buys him a chance to prevent the other agent from acquiring the good. This generates rents that can be captured by the seller. Assuming that the seller can commit to any mechanism proposed to the buyers, in the optimal mechanism, the seller commits to give the good for free to one agent if the other does not participate. In order to achieve entry of both bidders, the seller threatens them with their worst outcome if they do not participate, which is simply to suffer the externality with probability one. This reduces the lower bound of their payoff and therefore increases the rents that can be extracted from them. Moreover, the threat is costless, since it occurs only

4 Formally, the reservation utility of the agent is flat up to a given cutoff valuation and steeply increasing in the valuation after the cutoff. The author analyzes three cases. In case 1, the equilibrium utility binds at the bottom (on the flat part of the reservation utility). In case 2, the equilibrium utility binds at the top (on the steep part of the reservation utility); and in case 3, the equilibrium utility binds at the bottom and at the top.

5 When the reservation utility is always flat, the equilibrium utility binds at the bottom. This case is the same as case 1 in Carrillo (1998). When the reservation utility is steep, the equilibrium utility binds at the top. This case is the same as case 2 in Carrillo (1998).

6 We shall mention valuation-dependent positive externalities have also been studied in Brocas (2008) and Chen and Potipiti (2010). In such studies, the outside option is not type-dependent but countervailing incentives may arise through a tension in the incentives to report. Similar countervailing incentives may also arise under negative valuation-dependent externalities as shown in Brocas (2001). For the case of positive externalities, such countervailing incentives can be quite problematic. Chen and Potipiti (2010) study this case.

out-of-equilibrium. Naturally, this heavily relies on the commitment assumption.[7] In the rest of the analysis, we only need to make sure that participating guarantees bidders at least as much utility as their worst outside option.

We can invoke the revelation principle and restrict attention to direct mechanisms that are incentive compatible. A direct mechanism is characterized by the interim probability that agent i gets the good, $X_i(v_1, v_2)$ and the associated transfers $t_i(v_1, v_2)$. Let $u_i(v_i, v_i')$ be the expected *utility* of bidder i when he participates in the auction, his valuation is v_i, he announces v_i', and the other bidder discloses his true valuation v_j. We denote by $u_i(v_i) \equiv u_i(v_i, v_i)$ his expected utility under truthful revelation. We have:

$$u_i(v_i, v_i') = E_{v_j}\Big[v_i X_i(v_i', v_j) - \alpha_i(v_i, v_j) X_j(v_i', v_j) - t_i(v_i', v_j)\Big]. \quad (1)$$

Also, let $w_i(v_i)$ be the *reservation utility* of agent i, that is, his expected payoff when he does not participate in the auction, in which case the rival gets the good for sure. It is:

$$w_i(v_i) = -\alpha_a v_i - \alpha_b \int_{\underline{v}}^{\overline{v}} \frac{v_j}{\Delta} dv_j - \gamma. \quad (2)$$

The problem of the seller is to solve program $\mathcal{P}$:

$$\mathcal{P} : \max \int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{\overline{v}} \Big[t_1(v) + t_2(v)\Big] \frac{1}{\Delta^2} dv_1 dv_2$$

$$\text{s.t. } u_i(v_i) \geqslant u_i(v_i, v_i') \quad \forall\, i,\ v_i,\ v_i' \quad \text{(IC)}$$

$$u_i(v_i) \geqslant w_i(v_i) \quad \forall\, i,\ v_i \quad \text{(IR)}$$

$$X_i(v_i, v_j) \geqslant 0 \quad \forall\, i,\ v_i,\ v_j \quad (\text{F}_0)$$

$$X_1(v_i, v_j) + X_2(v_i, v_j) \leqslant 1 \quad \forall\, v_i,\ v_j \quad (\text{F}_1)$$

where (IC) is the incentive compatibility constraint, (IR) the individual rationality constraint and (F_0) and (F_1) are feasibility constraints. As usual (IC) is satisfied if and only if the two following conditions hold (see Appendix A.1 for details)

$$u_i(v_i) - u_i(v_i') = \int_{v_i'}^{v_i} E_{v_j}\Big[X_i(s, v_j) - \alpha_a X_j(s, v_j)\Big] ds$$
$$\forall\, i,\ v_i' \leqslant v_i \quad (\text{IC}_1)$$

$$E_{v_j}\Big[X_i(v_i', v_j) - \alpha_a X_j(v_i', v_j)\Big]$$
$$\leqslant E_{v_j}\Big[X_i(v_i, v_j) - \alpha_a X_j(v_i, v_j)\Big] \quad \forall\, i,\ v_i' \leqslant v_i \quad (\text{IC}_2)$$

where (IC_1) is the (first-order) local optimality condition which ensures that stating the true valuation $v_i' = v_i$ is a local optimum. (IC_2) is the (second-order) monotonicity condition and it ensures the convexity of the equilibrium utility, and therefore that the local optimum is a global maximum. Using (IC_1) and (2), we have:

$$\frac{d}{dv_i} u_i(v_i) = E_{v_j}[X_i(v)] - \alpha_a E_{v_j}[X_j(v)] \quad \text{and}$$
$$\frac{d}{dv_i} w_i(v_i) = -\alpha_a. \quad (3)$$

Informational rents are costly, and the seller wants to minimize $u_i(v_i) - w_i(v_i)$. There will be at least one agent who receives no rent at equilibrium, called a "binding type". Formally, it is a type $\hat{v}$ for which the (IR) constraint binds: $u_i(\hat{v}) = w_i(\hat{v})$.[8] From (3), we have:

$$\frac{d}{dv_i}(u_i(v_i) - w_i(v_i)) = \alpha_a \Big(1 - E_{v_j}[X_j(v)]\Big) + E_{v_j}[X_i(v)]. \quad (4)$$

Combining (3) and (4), we have four qualitatively different cases[9]: (i) when $\alpha_a = 0$, $w_i(v_i)$ is constant, $u_i(v_i)$ is increasing in v_i and $\hat{v} = \underline{v}$. This corresponds to the model analyzed by Jehiel et al. (1996), and shares the same technical aspects as the first case analyzed in Carrillo (1998) as well as the first case studied in Figueroa and Skreta (2009); (ii) when $\alpha_a > 0$, $w_i(v_i)$ is decreasing in v_i, $u_i(v_i)$ is not always monotonic in v_i and $\hat{v} = \underline{v}$. This case turns out to be technically similar to case (i)[10]; (iii) when $\alpha_a \leqslant -1$, $w_i(v_i)$ is increasing in v_i, $u_i(v_i)$ is increasing in v_i and $\hat{v} = \overline{v}$; This case shares common features with the second case analyzed in Carrillo (1998) as well as the second case studied in Figueroa and Skreta (2009)[11]; Last, (iv) when $\alpha_a \in (-1, 0)$, $w_i(v_i)$ is increasing in v_i, $u_i(v_i)$ is increasing in v_i and $\hat{v} \in [\underline{v}, \overline{v}]$. In this paper, we are interested in analyzing the mechanism design problem in case (iv). It is illustrated in Fig. 1.

The literature in incentive theory has fully analyzed optimal contracting under type-dependent reservation utilities in the single agent case (see e.g. Lewis and Sappington, 1989, Maggi and Rodriguez, 1995 and Jullien, 2000). To our knowledge the optimal mechanism when the binding type is interior has not been analyzed in a multi-agent setting except in simple limit cases. The authors either restricted attention to corner binding types (as in the first two cases in Carrillo (1998) and Figueroa and Skreta (2009), or Brocas (2013)) or treated cases boiling down to single agent problems (as in the third case in Figueroa and Skreta (2009)). Our objective is to offer a complementary analysis. Let $H(v_i) = E_{v_j}[X_i(v_i, v_j) - \alpha_a X_j(v_i, v_j)]$ from now on.

3. Optimal mechanism

3.1. The optimization program of the seller

The seller chooses a vector of interim probabilities such that the incentive compatibility constraints ((IC_1)–(IC_2)), the individual rationality constraint (IR) and the feasibility constraints ((F_0)–(F_1)) are satisfied. Given informational rents are costly to her, she selects an allocation rule such that (IR) is binding.

Lemma 1. *For any mechanism A with vector of interim probabilities $X = (X_1(v), X_2(v))$ satisfying (IC_1)–(IC_2)–(F_0)–(F_1), the set of binding types is an interval $\hat{V}_i(A) = [\hat{v}_i^a(A), \hat{v}_i^b(A)]$.*

Proof. See Appendix A.2.

This results from the fact that the equilibrium utility must be weakly increasing and convex while the reservation utility is an increasing linear function (see Fig. 1). Also $u_i(v_i) > w_i(v_i)$ for all $v_i \notin \hat{V}_i(A)$. In the absence of adequate incentives, an agent with a valuation $v_i < \hat{v}_i^a(A)$ will over-state his type and an agent with a valuation $v_i > \hat{v}_i^b(A)$ will under-state his type. To induce truthful revelation, informational rents have to be decreasing up to $\hat{v}_i^a(A)$ and increasing after $\hat{v}_i^b(A)$, as depicted in Fig. 1 (which is drawn for

7 Although standard in the literature on auctions with externalities (see e.g. Carrillo, 1998, Jehiel et al., 1996, 1999 etc.) and sometimes not even discussed, this assumption is strong. If an agent does not show up, the seller will have ex-post incentives to conduct the auction with only one bidder rather than give him the good for free. In Brocas (2003), we show that when this assumption is relaxed, there is a coordination problem in the behavior of agents that gives rise to multiple equilibria.

8 At this stage, we cannot establish whether the binding type is unique or not.

9 Recall that, given (F_1), we have $1 - E_{v_j}[X_j(v)] \geqslant E_{v_j}[X_i(v)]$.

10 However, it delivers novel economic implications due to the fact that the equilibrium utility may not be monotonic (the r.h.s. of (IC_1) is not necessarily positive). At equilibrium, intermediate types will receive less utility compared to low and high types. See Brocas (2013) for the complete solution of this case.

11 It also delivers new economic implications. In particular, the bidder with the lowest valuation gets the good and the auction must be implemented with price ceilings rather than reserve prices. See Brocas (2013).

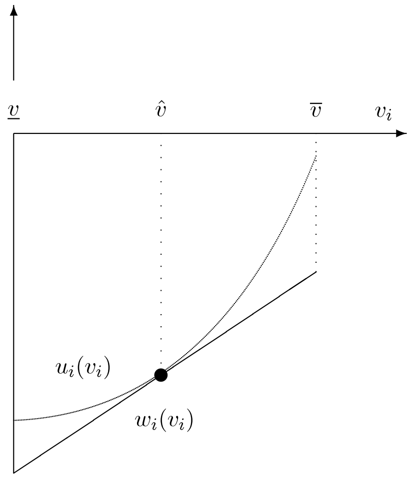

Fig. 1. Equilibrium and reservation utilities.

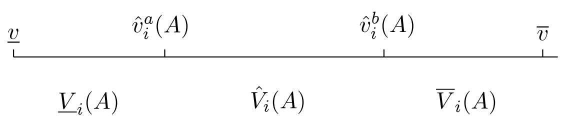

Fig. 2. Relevant sets of valuations.

the limit case where $\hat{v}_i^a(A) = \hat{v}_i^b(A)$). Using (IC$_1$), the equilibrium rent is then

$$u_i(v_i) = \begin{cases} w_i(\hat{v}_i^a(A)) - \int_{v_i}^{\hat{v}_i^a(A)} H(s)ds & v_i < \hat{v}_i^a(A) \\ w_i(v_i) & v_i \in \hat{V}_i(A) \\ w_i(\hat{v}_i^b(A)) + \int_{\hat{v}_i^b(A)}^{v_i} H(s)ds & v_i > \hat{v}_i^b(A) \end{cases} \tag{5}$$

and the necessary condition for (IR) to bind on $\hat{V}_i(A)$ is

$$H(v_i) = -\alpha_a \quad \forall\, v_i \in \hat{V}_i(A),\ \forall\, i. \tag{$\widehat{\text{IR}}$}$$

We will call the 'lower subset' the set of values below the set of binding types $\underline{V}_i(A) = [\underline{v}, \hat{v}_i^a(A))$ and the 'upper subset' the set of values above it, $\overline{V}_i(A) = (\hat{v}_i^b(A), \overline{v}]$. These are illustrated in Fig. 2.

Overall, we are looking for a vector of interim probabilities satisfying constraints (IC$_2$)–(F$_0$)–(F$_1$), generating a set of binding types satisfying Eq. ($\widehat{\text{IR}}$) and a rent profile satisfying Eq. (5). Among those, the seller selects the one that provides highest revenue. From now on and to economize on notations, we will omit the dependence of those sets on the mechanism A, although the reader should keep in mind their mechanism-dependency. Using (1) and (5), the seller's optimization program $\mathcal{P}$ is now equivalent to program $\hat{\mathcal{P}}$:

$$\hat{\mathcal{P}} : \max \sum_I \sum_J \int_I \int_J \Big[X_i(v)\pi_i^{IJ}(v_i, v_j) + X_j(v)\pi_j^{IJ}(v_i, v_j)\Big] \times \frac{1}{\Delta^2}dv - \sum_i \Bigg(\int_{\underline{v}}^{\hat{v}_i^a} w_i(\hat{v}_i^a)\frac{1}{\Delta}dv + \int_{\hat{v}_i^a}^{\hat{v}_i^b} w_i(v)\frac{1}{\Delta}dv + \int_{\hat{v}_i^b}^{\overline{v}} w_i(\hat{v}_i^b)\frac{1}{\Delta}dv\Bigg)$$

s.t. (IC$_2$)–(F$_0$)–(F$_1$)–($\widehat{\text{IR}}$)

where $\pi_i^{IJ}(v_i, v_j)$ is the virtual surplus of selling to agent i when $v_i \in I$ and $v_j \in J$,

$$\pi_i^{IJ}(v_i, v_j) = v_i - \alpha_j(v) - (\overline{v} - v_i)1_{I=\overline{V}_i} + (v_i - \underline{v})1_{I=\underline{V}_i} + \alpha_a(\overline{v} - v_j)1_{J=\overline{V}_j} - \alpha_a(v_j - \underline{v})1_{J=\underline{V}_j},$$

$I \in \{\underline{V}_i, \hat{V}_i, \overline{V}_i\}$ and $J \in \{\underline{V}_j, \hat{V}_j, \overline{V}_j\}$. There are 9 possible combinations of I and J, yielding 9 possible virtual surpluses. Each of them represents the net surplus that the auctioneer can extract by selling the good to agent i rather than keeping it, adjusted for the informational rents that she is obliged to grant due to the asymmetry of information vis-a-vis bidders. When externalities are present, agents are willing to pay to prevent the allocation to their rival. Therefore, under complete information, the seller can extract v_i from agent i by selling the good to i or she can extract $\alpha_j(v)$ from agent j if she keeps the good. The net surplus of the sale to i is therefore $v_i - \alpha_j(v)$. Under asymmetric information, the seller leaves extra rents reflected in the extra terms. Note that the distortion due to informational rents acts differently depending on whether a type lies in the lower subset or the upper subset of the set of binding types.[12] Last, given the interdependency between types and externalities, increasing the probability of allocating the good to i also affects the rents to be granted to j.

Assumption 3. $\alpha_b \leqslant 0$.

This assumption guarantees that the virtual surpluses are increasing in v_i and makes the problem regular in the terminology of Myerson (1981).

3.2. The effect of countervailing incentives on the multi-agent problem

According to program $\hat{\mathcal{P}}$, the seller must choose an allocation rule $(X_i(v), X_j(v))$ that satisfies (IC$_2$), (F$_0$) and (F$_1$) and generates a set of binding types that satisfies ($\widehat{\text{IR}}$). Among all those allocations, the optimal mechanism is the one that generates the highest revenue. A classic procedure to solve for this type of problem is to relax the constraints to obtain an unconstrained solution and show that the solution satisfies the constraints ex-post. In some cases, (IC$_2$) is violated, and procedures have been developed to restore it. Compared to the earlier literature, our problem is different because (IC$_2$) is not the only problematic constraint. We also need to satisfy ($\widehat{\text{IR}}$). To better understand the impact of each of these constraints and their combination on the optimization problem, we will first consider a relaxed problem and determine which constraint is or is not satisfied.

Suppose the seller considers mechanisms that satisfy (F$_0$) and (F$_1$) and restricts attention to those that generate the highest virtual surplus on each of the 9 regions provided they exist. We will show that such a mechanism does not satisfy the remaining constraints (IC$_2$) and ($\widehat{\text{IR}}$) generically, and we will study the implications of those violations. To do so, let us introduce a relaxed optimization program which is as if the seller guesses arbitrary sets $\hat{V}_i$ and $\hat{V}_j$ and looks for the best mechanism that satisfies (F$_0$) and (F$_1$), given this guess:

$$\mathcal{P}^{UNC}(\hat{V}_i, \hat{V}_j) : \max \sum_I \sum_J \int_I \int_J \Big[X_i(v)\pi_i^{IJ}(v_i, v_j) + X_j(v)\pi_j^{IJ}(v_i, v_j)\Big]\frac{1}{\Delta^2}dv - \sum_i \Bigg(\int_{\underline{v}}^{\hat{v}_i^a} w_i(\hat{v}_i^a)\frac{1}{\Delta}dv + \int_{\hat{v}_i^a}^{\hat{v}_i^b} w_i(v)\frac{1}{\Delta}dv + \int_{\hat{v}_i^b}^{\overline{v}} w_i(\hat{v}_i^b)\frac{1}{\Delta}dv\Bigg)$$

s.t. (F$_0$)–(F$_1$)

[12] By increasing the probability of allocating the good to agent i at a point v_i in the lower subset, the seller must grant extra rents to all types below v_i (in proportion $(v_i - \underline{v})/\Delta$). Conversely, by increasing the probability of allocating the good to agent i at a point v_i in the upper subset, the seller must grant extra rents to all types above v_i (in proportion $(\overline{v} - v_i)/\Delta$).

for all $I \in \{\underline{V}_i, \hat{V}_i, \overline{V}_i\}$ and $J \in \{\underline{V}_j, \hat{V}_j, \overline{V}_j\}$. Let $r_i^{IJ}(v_j) = \min\{v_i \in I | \pi^{IJ}(v_i, v_j) \geqslant 0\}$ and $h^{IJ}(v_j) = \min\{v_i \in I | \pi_i^{IJ}(v_i, v_j) \geqslant \pi_j^{IJ}(v_i, v_j)\}$. Given our assumptions, $r_i^{IJ}(v_j)$ is decreasing in v_j and $h^{IJ}(v_j)$ is increasing in v_j (see Appendix A.3 for details).

Lemma 2. *The solution of* $\mathcal{P}^{UNC}(\hat{V}_i, \hat{V}_j)$ *is the mechanism* $A^{UNC}(\hat{V}_i, \hat{V}_j)$ *with the following properties.*

(i) *The allocation rule is such that in each IJ*

$$X_i^{UNC}(v_i, v_j) = \begin{cases} 1 & \text{if } v_i > \max\{r_i^{IJ}(v_j), h^{IJ}(v_j)\} \\ 0 & \text{otherwise} \end{cases}$$

(ii) $A^{UNC}(\hat{V}_i, \hat{V}_j)$ *does not satisfy* (IC_2) *at* $\hat{v}_i^a$ *and* $\hat{v}_i^b$;

(iii) $A^{UNC}(\hat{V}_i, \hat{V}_j)$ *does not satisfy* ($\widehat{IR}$).

Proof. See Appendix A.3.

The solution of $\mathcal{P}^{UNC}(\hat{V}_i, \hat{V}_j)$ is obtained in the standard way. The good is allocated to the agent that generates the highest virtual surplus provided it is positive (point (i)). This allocation is depicted in Fig. 3 which is drawn for the case $\hat{v}_i^a = \hat{v}_i^b = \hat{v}$ for all i. As a consequence, we have only 4 regions. In each region, the downward slopping curves represent the reserve prices faced by each agent (these are the curves of $r_i^{IJ}(v_j)$) and the upwards slopping curves separate the region in which it is best to give the good to agent i rather than j from the region in which the opposite allocation is optimal (these are the curves of $h^{IJ}(v_j)$).

However, to be a possible solution of $\hat{\mathcal{P}}$, the vector of probabilities in $A^{UNC}(\hat{V}_i, \hat{V}_j)$ must at least (a) be such that (IC_2) is satisfied, (b) generate a unique set of binding types satisfying ($\widehat{IR}$), and (c) this set of binding types must coincide with the guessed $\hat{V}_i$. Lemma 2 shows these requirements are not met. First, property (a) is violated. This is the case because the problem is not regular everywhere. Namely (IC_2) is satisfied for all $v_i < \hat{v}_i^a$, $v_i \in (\hat{v}_i^a, \hat{v}_i^b)$ and $v_i > \hat{v}_i^b$ but the function $H(v_i)$ admits downwards jumps at $\hat{v}_i^a$ and $\hat{v}_i^b$ that violates (IC_2). This is illustrated in Fig. 4. Second, properties (b) and (c) do not hold. The discontinuities at points $\hat{v}_i^a$ and $\hat{v}_i^b$ generate discontinuities in $H(v_i)$ and ($\widehat{IR}$) is not satisfied for free. The set of points for which the derivative of the utility profile coincides with that of the outside option is not $\hat{V}_i$. It may not be a subset either, as illustrated in Fig. 4 that features three disjoint points for which $H(v_i) = -\alpha_a$. Overall, it is not possible to find a mechanism that maximizes the revenue of the seller pointwise and satisfies the constraints.

Compared to the existing literature, the violation of (IC_2) shares common features with other studies but the violation of ($\widehat{IR}$) is a new violation. Given that the two constraints are interrelated, it is not possible to use standard techniques to restore them starting from $\mathcal{P}^{UNC}(\hat{V}_i, \hat{V}_j)$. We shall spend a few paragraphs explaining why.

The violation of (IC_2) is reminiscent of the previous literature on countervailing incentives in the single agent case (see for instance Maggi and Rodriguez, 1995), and it relates to a tension between the way the seller wants to solve the trade-off between rents and efficiency when types lie in the upper and lower sets. To see this, note that the rent $u_i(v_i) - w_i(v_i)$ is decreasing in the lower subset and increasing in the higher subset. In the lower subset, the seller would like to decrease the rent by making the slope of the rent less negative (and make the expected utility come closer to the outside option). This pushes her to increase the probability of allocating the good compared to the full information setting. In the upper subset however, the seller would like to decrease the rent by making the slope of the rent less positive. This pushes her to decrease the probability of allocating the good compared to the full information setting. Overall, the seller would like to allocate the good relatively more often in the lower subset compared to the interior set of binding types, and also relatively more often in the set of binding types

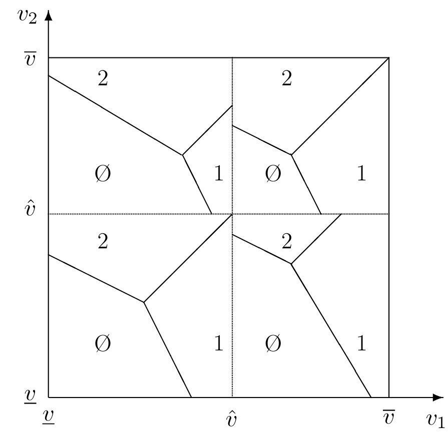

Fig. 3. Allocation A^{UNC}.

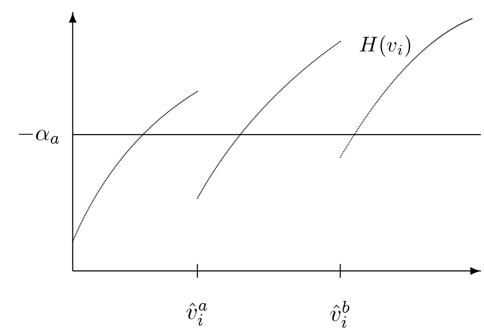

Fig. 4. Violations of (IC_2) and ($\widehat{IR}$).

compared to the upper subset. Therefore, there are discontinuities at $\hat{v}_i^a$ and $\hat{v}_i^b$.

Guesnerie and Laffont (1984) examine violations of (IC_2) in single-agent contracting problems. When the unconstrained allocation violates (IC_2), the solution is to construct an allocation such that bunching occurs on intervals containing the points at which (IC_2) is violated, and that coincides with the unconstrained allocation everywhere else. In multi-agent contracting problems, Myerson (1981) restores (IC_2) by replacing the virtual surplus with an *ironed virtual surplus* that coincides with the virtual surplus except on some intervals on which both surpluses are equal in expectation.

Our problem differs in several respects. First, our monotonicity condition (IC_2) requires some form of monotonicity of the *ex-ante* allocation.[13] However, there is not a unique or obvious way of distorting the *ex-post* allocation (X_i, X_j) to restore (IC_2). It could be done by distorting the reserve price of agent i, that of agent j or the decision rule to allocate to i versus j. This difficulty is not present in the single-agent setting (e.g. Guesnerie and Laffont, 1984) because it is as if the ex-ante and ex-post allocations coincide (see Appendix B for an example of a single-agent allocation problem related to ours in which the standard approach applies). Second, the virtual surpluses have the desired properties, making the problem regular (except at $\hat{v}_i^a$ and $\hat{v}_i^b$). The issue is that regularity is not enough in our case: the monotonicity of the virtual surplus does not guarantee the monotonicity of $H(v_i)$. Myerson (1981) ironing technique is not designed for our problem. Third, the monotonicity conditions required to induce both agents to report truthfully are interrelated. This complication is absent from all previous frameworks. Suppose that we modify the ex-post allocation of agent i in the neighborhood of $\hat{v}_i^a$. That is, we distort the probability of allocating the good to agent i for some values v_j of agent j. This in turn

[13] $H(v_i)$ is an expectation over the ex-post probabilities X_i and X_j.

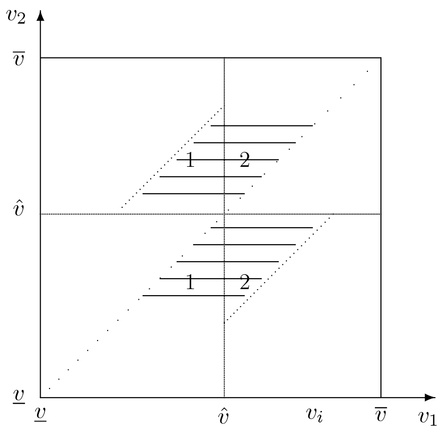

Fig. 5a. Region of disagreement $\hat{W}_1$ (striped area).

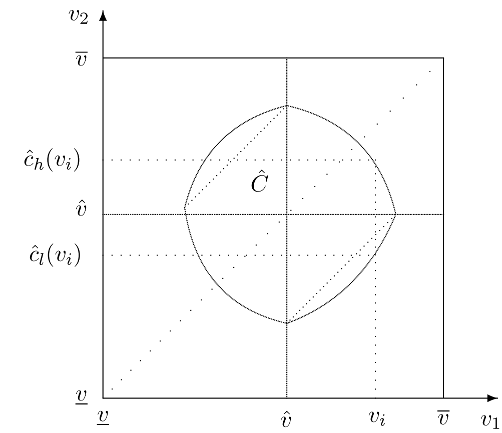

Fig. 5b. Construction of an arbitrary $\hat{C}$.

affects the probability of allocating the good to j for those v_j. Therefore, attempting to restore (IC$_2$) vis-à-vis agent i modifies the shape of the function $H(v_j)$, which may now not be monotonic where it used to be. In other words, restoring (IC$_2$) where it is violated may affect the mechanism at values at which (IC$_2$) is satisfied. This results from the interdependency of the allocation rules. Last, it is necessary to restore (IC$_2$) and $(\widehat{\text{IR}})$. Those two constraints are related, and the latter imposes a restriction on monotonicity: $H(v_i)$ must be constant exactly on the set of binding types.[14]

3.3. *Optimal mechanism with countervailing incentives*

In this section we develop a procedure to identify some properties of the optimal ex-post allocation. As noted previously, (IC$_2$) and $(\widehat{\text{IR}})$ cannot be neglected and checked ex post. We therefore need to construct the set of mechanisms that satisfy those constraints and select the optimal mechanism from that set.

For each interval $\hat{V}_i$, there exists a set of all mechanisms that satisfy (IC$_2$) and such that $H(v_i) = -\alpha_a$ for all $v_i \in \hat{V}_i$. Whenever the set of such mechanisms is non empty, there exists one mechanism that yields the highest revenue to the seller. Intuitively, such mechanism will entail allocation rules 'as close as possible' to property (i) in Lemma 2. Given this is true for all intervals $\hat{V}_i$, there exists a set of all those mechanisms that yield the highest revenues. The optimal mechanism is the one that provides the highest revenue among those.[15]

Let us focus first on the mechanism that yields highest revenue for a given possible set of binding types $\hat{V}_i$. Let us fix $\hat{V}_i$ and $\hat{V}_j$. Note first that it would be a priori better to allocate to one agent to the right of $\hat{v}_i^a$ and to the other to the left of that point. This is the case because the virtual surpluses are highest if the item is allocated to the agent with the highest valuation on one side of $\hat{v}_i^a$ and to the agent with the lowest valuation on the other of $\hat{v}_i^a$. The same holds for $\hat{v}_i^b$ and this also applies to agent j (see Fig. 3). Recall that maximizing the virtual surplus in that region is precisely what causes violations of (IC$_2$) (as emphasized in Lemma 2). Let $z_i(v)$ (respectively $z_j(v)$) be the virtual surplus derived from selling to i (respectively j) at point v and define:

$$Y(v) = \begin{cases} i & z_i(v) \geqslant z_j(v) \\ j & z_i(v) < z_j(v) \end{cases}$$

$Y(v)$ returns the identity of the agent generating the highest virtual surplus. For all $\hat{v}_i^k$, there exists v_j, $v_i^{k^-}(v_j) < \hat{v}_i^k$ and $v_i^{k^+}(v_j) > \hat{v}_i^k$ such that

$$\begin{aligned} v_i^{k^-}(v_j) &= \min\Big\{ v_i | Y(v_i, v_j) = Y(v_i', v_j) = Y(\hat{v}_i^k - \epsilon, v_j), \\ &\qquad \forall\, v_i' \in [v_i, \hat{v}_i^k - \epsilon],\ \epsilon > 0 \text{ and } \epsilon \to 0 \Big\} \\ v_i^{k^+}(v_j) &= \max\Big\{ v_i | Y(v_i, v_j) = Y(v_i', v_j) = Y(\hat{v}_i^k + \epsilon, v_j), \\ &\qquad \forall\, v_i' \in [\hat{v}_i^k + \epsilon, v_i],\ \epsilon > 0 \text{ and } \epsilon \to 0 \Big\}. \end{aligned}$$

Note that by construction, for all $v_i \in (v_i^{k^-}(v_j), \hat{v}_i^k)$ and all $v_i' \in (\hat{v}_i^k, v_i^{k^+}(v_j))$, $Y(v_i, v_j) \neq Y(v_i', v_j)$.[16] Let

$\hat{W}_i^k = \{(v_i, v_j) | v_i \in (v_i^{k^-}(v_j), v_i^{k^+}(v_j))\} \quad k = \{a, b\}$

the set in $\mathbb{R}^2$ of all points in the neighborhood of $\hat{v}_i^k$ such that the allocation to the left of $\hat{v}_i^k$ does not agree with the allocation to its right. Consider $\hat{W}_{12} = \hat{W}_1^a \bigcup \hat{W}_2^a \bigcup \hat{W}_1^b \bigcup \hat{W}_2^b$. This set is illustrated in Fig. 5a for the case $\hat{V}_i = \hat{v}$.[17] Finally, let $\hat{C}$ be the convex hull of $\hat{W}_{12}$ as illustrated in Fig. 5b, again for the case $\hat{V}_i = \hat{v}$. By construction, $\hat{C}$ contains all the points that generate violations of (IC$_2$) due to inconsistent allocations to agents 1 and 2 in $A^{UNC}(\hat{V}_i, \hat{V}_j)$. Therefore, the allocation depicted in property (i) in Lemma 2 has to be distorted in $\hat{C}$. By construction, there is no tension in the complement of $\hat{C}$. In particular, and other things being equal, it is preferred to allocate the item to agent i when $v_i \geqslant v_j$ and to agent j when $v_i < v_j$.

14 A better characterization cannot be obtained even if we limit ourselves to the ex-ante allocation (characterize $E_{v_j}X_i(v)$). Such an approach has been used for example in Maskin and Riley (1984) in a different setting. The authors exploit the symmetry of the problem to rewrite it as a single agent problem (if an optimal mechanism A is asymmetric, then its counterpart A' is also optimal and therefore the symmetric mechanism that consists in implementing A with probability μ and A' with probability $1-\mu$ is also optimal). In their case, this allows to transform the problem into a standard optimal control problem and they can characterize properties of the optimal ex-ante allocation. In our case, however, the correlations between externalities and valuations prevent us from rewriting the problem in a standard form. Technically, we need to obtain an objective function that depends on the probability of allocating the good to the agent, but this always fails: the objective function depends also on the probability of allocating the good to the rival. To see this, assume $\alpha_b = 0$ to simplify. Let $P_i(v_i) = E_{v_j}X_i(v_i, v_j)$ and $Q_j(v_i) = E_{v_j}X_j(v_i, v_j)$, the expected transfer paid by agent i is $T(v_i) = E_{v_j}t_i(v_i, v_j) = P_i(v_i) - (\alpha_a v_i + \gamma)Q_j(v_i) - u_i(v_i)$. Noting that probabilities must satisfy $\int_{\underline{v}}^{\overline{v}} Q_j(v_i)dv_i = \int_{\underline{v}}^{\overline{v}} P_j(v_j)dv_j$ is enough to obtain the expected transfer as a function of $P_i(v_i)$ only when $\alpha_a \neq 0$.

15 Note that this iterative procedure compares all possible mechanisms that satisfy the constraints and eliminates iteratively those that are dominated.

16 For all $v_j \in J$ with $J \in \{\underline{V}_j, \hat{V}_j, \overline{V}_j\}$, the points $v_i^{a^-}(v_j)$ coincide with $h^{\underline{V}_i J}(v_j)$. The points $v_i^{a^+}(v_j)$ coincide with $h^{\overline{V}_i J}(v_j)$ if $\hat{v}_i^a = \hat{v}_i^b$ and with $h^{\hat{V}_i J}(v_j)$ if $\hat{v}_i^a \neq \hat{v}_i^b$. Similarly, for all $v_j \in J$ with $J \in \{\underline{V}_j, \hat{V}_j, \overline{V}_j\}$, the points $v_i^{b^+}(v_j)$ coincide with $h^{\overline{V}_i J}(v_j)$. The points $v_i^{b^-}(v_j)$ coincide with $h^{\underline{V}_i J}(v_j)$ if $\hat{v}_i^a = \hat{v}_i^b$ and with $h^{\hat{V}_i J}(v_j)$ if $\hat{v}_i^a \neq \hat{v}_i^b$.

17 Note that the upwards slopping dotted lines represent $h^{IJ}(v_j)$ which coincide with $v_i^{k^-}(v_j)$, $v_i^{k^+}(v_j)$ (and, by construction, the inverse functions of $v_j^{k^-}(v_i)$, $v_j^{k^+}(v_i)$).

With this in mind, we will fix an allocation in $\hat{C}$ and we will determine how the allocation should be completed in the complement of $\hat{C}$ to satisfy (IC$_2$) and $(\widehat{\text{IR}})$. Among the resulting allocations, we will describe the one that provides highest revenue. Conversely, we will fix an allocation in the complement of $\hat{C}$ and we will determine how the allocation should be completed in $\hat{C}$ to satisfy (IC$_2$) and $(\widehat{\text{IR}})$, then describe the allocation that provides highest revenue. We look for an equilibrium, that is for an allocation such that the allocation in $\hat{C}$ is optimal given the allocation in its complement and the allocation in the complement is optimal given the allocation in $\hat{C}$. Last, and as noted earlier, the optimal mechanism is the best of all those equilibrium allocations revenue-wise.

We shall first make sure that the allocation in $\hat{C}$ does not violate $(\widehat{\text{IR}})$, that is we have "enough room" to construct an allocation that satisfies the constraint. For every v_i, there exists an interval $\hat{C}(v_i) = [\min\{\hat{c}_l(v_i), v_i\}, \max\{\hat{c}_h(v_i), v_i\}]$ of values v_j such that $(v_i, v_j) \in \hat{C}$ (see Fig. 4).

Definition 1. An allocation is feasible if for all i, j, $\frac{1}{\Delta}\int_{\hat{C}(v_i)}(X_i(v_i, v_j) - \alpha_a X_j(v_i, v_j))dv_j < -\alpha_a$ for all $v_i \leqslant \hat{v}_i^a$, and $\frac{1}{\Delta}\int_{\hat{C}(v_i)}\Big(X_i(v_i, v_j) - \alpha_a X_j(v_i, v_j)\Big)dv_j \leqslant -\alpha_a$ for all $v_i \in (\hat{v}_i^a, \hat{v}_i^b)$.

Non feasible allocations cannot be optimal and we restrict to feasible ones.

Proposition 1. *The optimal mechanism is characterized by an allocation rule* $(\hat{X}_i^*, \hat{X}_j^*)$, *and associated sets* $\hat{V}_i^*, \hat{V}_j^*, \underline{V}_i^*, \underline{V}_j^*, \overline{V}_i^*, \overline{V}_j^*$ *and* $\hat{C}^*$, *with the following properties:*

1. *The allocation in the complement of* $\hat{C}^*$ *is such that for all* $v_i > v_j$:
   (i) *Agent j never obtains the good.*
   (ii) *For all* $v_i \in \hat{V}_i^*$, *there exists a point* $m(v_i) \in \Big(\underline{v}, \hat{c}_l(v_i)\Big)$ *such that*
   $$\hat{X}_i^*(v_i, v_j) = \begin{cases} 1 & \text{if } v_j \in (m(v_i), \hat{c}_l(v_i)) \\ 0 & \text{if } v_j < m(v_i) \end{cases}$$
   and set in such a way that $(\widehat{\text{IR}})$ *is satisfied.*
   (iii) *For all* $v_i \in \underline{V}_i^*$ *and such that* $v_i < r_i^{\underline{V_i^*V_j^*}}(v_j)$, $\hat{X}_i^*(v_i, v_j) = 0$ *unless* (IC$_2$) *is violated.*
   (iv) *For all* $v_i \in \overline{V}_i^*$ *and such that* $v_i > r_i^{\overline{V_i^*V_j^*}}(v_j)$, $\hat{X}_i^*(v_i, v_j) = 1$.
   (v) *Everywhere else,* $\hat{X}_i^*(v_i, v_j) \in \{0, 1\}$.
2. *The allocation in* $\hat{C}^*$ *is such that*
   (i) *If* $v_i > v_j$ *and* $\pi_i^{IJ}(v) > \pi_j^{IJ}(v)$, *then* $\hat{X}_j^*(v_i, v_j) = 0$ *and* $\hat{X}_i^*(v_i, v_j) \in \{0, 1\}$.
   (ii) *If* $v_i > v_j$ *and* $\pi_i^{IJ}(v) < \pi_j^{IJ}(v)$, *then either the good is not allocated or*
   $\hat{X}_i^*(v_i, v_j) = b(v_i, v_j)$ *and* $\hat{X}_j^*(v_i, v_j) = 1 - b(v_i, v_j)$ *where* $b(v_i, v_j)$ *is a probability.*
3. *The sets of binding types* $\hat{V}_i^* \subset (\underline{v}, \overline{v})$ *and the complement of* $\hat{C}^*$ *is never empty.*

Proof. See Appendix A.4.

We will first concentrate on the allocation on the complement of the equilibrium $\hat{C}^*$. The reader shall keep in mind that, absent the constraints, the revenue is maximized if the allocation satisfies (i) in Lemma 2. In particular, it is best to allocate to *i* if it is above a reserve price. Also the surplus derived from selling to *i* increases in *j*'s valuation: for a given v_i, the seller extracts more rents from *i* when v_j is high. Moreover, it is more profitable to sell the good to the agent with the highest valuation anywhere outside any arbitrary $\hat{C}$.

Keeping this in mind, the optimal allocation on the complement of $\hat{C}^*$ has intuitive properties illustrated in Fig. 6a. To simplify the exposition, let us concentrate on the region below the 45° degree line (the argument is symmetric for the region above). First and as expected, the optimal mechanism allocates the good to agent *i* who has the highest valuation, or to no one (item (i)). Second, for every $\hat{C}$, every feasible allocation in $\hat{C}$, and every allocation above the 45° line, it is always possible to satisfy $(\widehat{\text{IR}})$. Indeed, for valuations $v_i \in \hat{V}_i$ the seller can allocate the good to *i* at points (v_i, v_j) making sure that $H(v_i) = -\alpha_a$. Given the surplus extracted from selling to *i* is increasing in v_j, the seller prefers to allocate the good to *i* with probability 1 when the highest possible v_j realize. Therefore, *i* receives the object when $v_j > m(v_i)$ and $m(v_i)$ is computed to satisfy $(\widehat{\text{IR}})$. This logic applies to all arbitrary $\hat{C}$ and a fortiori to the equilibrium allocation (item (ii)). Third, for any v_i in the lower subset, the virtual surplus is negative if agent *i* receives the good when $v_i < r_i^{\underline{V_iV_j}}(v_j)$. However, the allocation in that region must also be such that (IC$_2$) is satisfied and $H(v_i) < -\alpha_a$. Even though the seller finds it optimal to not allocate to *i* when v_i is too small, some non profitable trades may be implemented to satisfy those constraints (item (iii)). Similarly, for any v_i in the upper subset, the virtual surplus is maximized if agent *i* receives the good provided $v_i > r_i^{\overline{V_iV_j}}(v_j)$. In that region, increasing the probability of allocating the good to *i* does not conflict with (IC$_2$). However, it may be necessary to give the good to *i* more often than optimal to satisfy $H(v_i) > -\alpha_a$ (item (iv)). Last, when $v_i \in (r_i^{\underline{V_iV_j}}(v_j), r_i^{\overline{V_iV_j}}(v_j))$, the good is either allocated to *i* or not allocated at all (part (v)). Preference is given to points that generate the highest surplus levels (and involve the highest v_j) and the decision to allocate or not is constrained by the requirement of the constraints. Overall, and as shown in Fig. 6a, the allocation consists in allocating to agent *i* except when low values of v_j realize. Again, this is true for any arbitrary $\hat{C}$ and any strategy in $\hat{C}$ and the property must hold in the optimal mechanism as well.

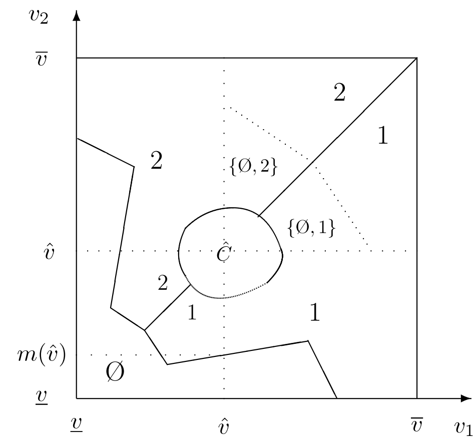

Fig. 6a. Optimal allocation in the complement of $\hat{C}$ ({∅, *i*} means that the good is sold to *i* or to none).

The allocation in $\hat{C}^*$ follows the same general principles (item 2) and is depicted in Fig. 6b. The main difference is that, depending on the region *IJ* we consider, the revenue is not always maximized by allocating to the agent with the highest valuation. Other things being equal, the seller wants to allocate the good to either agent as long as it is profitable and does not conflict with the constraints. However, and as seen in Lemma 2, following the surplus maximizing rule is the source of discontinuities. Therefore, it is necessary to distort the allocation. Let us concentrate again on the valuations lying below the 45° line, and fix an allocation elsewhere. The problem of the seller is to allocate the good to make sure $H(v_i)$ has the required property in each region. Whenever it is necessary to allocate the good more often than what is profitable ($H(v_i)$ must be increased), the seller allocates the good to *i* or *j* at points associated with the smallest 'loss'. Whenever it is necessary to allocate the good less often than profitable ($H(v_i)$ must be decreased), the seller discriminates against pairs of valuations associated with the

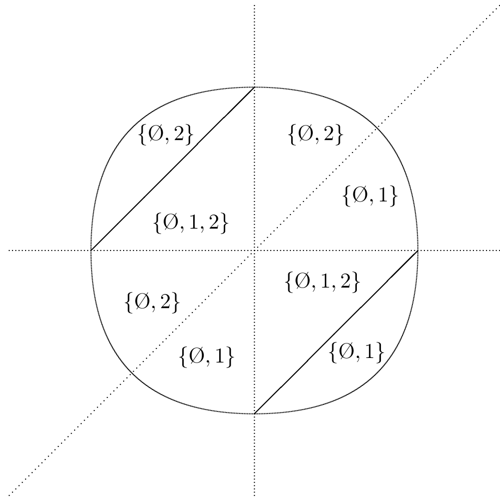

Fig. 6b. Optimal allocation in $\hat{C}$ ($\{\emptyset, i, j\}$ means that the good is sold to i, j or to none).

smallest surplus. Moreover, decreasing the overall probability of allocating the item to i has a relatively bigger impact on $H(v_i)$ and the seller tends to distort this allocation more often to satisfy the constraints. At equilibrium, agent i may receive the good with some probability when it is optimal to allocate it to j. Precisely, if $v_i > v_j$ and the surplus obtained by selling to i is greater, the seller chooses to either allocate to i or to nobody (item (i)). However, if the surplus obtained by selling to j is greater, the seller may randomize between the two agents (item (ii)).

Last, the equilibrium $\hat{V}_i^*$ is never the full support (item 3) and therefore the complement of $\hat{C}^*$ is never empty. Suppose by contradiction it is. The unconstrained allocation corresponding to a guessed $\hat{V}_i = \hat{V}_j = [\underline{v}, \overline{v}]$ requires to allocate the good to i whenever $v_i > v_j$ and provided $v_i > r_i^{\hat{V}_i\hat{V}_j}(v_j)$. This mechanism is of course not optimal as it cannot satisfy $(\widehat{\text{IR}})$ which requires $H(v_i) = -\alpha_a$ for all v_i. Restoring the constraint requires to allocate the good more often than optimal for low values of v_i and less often than optimal for high values of v_i. As this is costly to the seller, it creates a motive for reducing the set of binding types.

It can be seen from Figs. 3, 6a and 6b that discontinuities and violations in $A^{UNC}(\hat{V}_i, \hat{V}_j)$ are partly due to the fact that the revenue maximizing rule induces to allocate too often to agent 2 when $v_1 > v_2$ and to agent 1 when $v_2 > v_1$. This creates a misalignment of incentives. In the optimal mechanism, the agent with the highest valuation obtains the good more often, that is often enough to be induced to reveal truthfully.

4. Implications and concluding remarks

This article studies the allocation mechanism of a single item in the presence of type-dependent externalities between two bidders. The type-dependency introduces countervailing incentives. Therefore, the optimal allocation must be sometimes such that the equilibrium utility is equal to the reservation utility on an interior subset of types. I have shown that this problem is technically different from the one analyzed in related single agent settings because the seller must manage two constraints that conflict with her objective. First, truth-telling requires the ex-ante allocation to satisfy a non-trivial monotonicity condition. Second, the allocation must be such that some types receive no rent. I have provided a procedure to characterize the main properties of the ex-post allocation.

The optimal mechanism has two novel and interesting properties. All relate to the *competition* between two agents facing countervailing incentives. First, the ex-post allocation of the good is contingent on whether each of the two types lies in the lower set, the set of binding types or the upper set. As a consequence, a bidder does not face a single reserve price but a family of reserve prices. This is the case because the trade-off between efficiency and rents is solved differently if types lie in either of these sets. The ability to extract rents varies as a function of the relative "strength" of the bidders, and the allocation rule must be tailored to it.

Second, the seller sometimes refrains from allocating the good while it would be profitable in order to make sure the constraints are satisfied. Interestingly, she may randomize between the two agents. Tailoring the rules to solve optimally the trade-off between rent extraction and efficiency creates a misalignment of incentives. Intuitively, in this competitive setting, solving the trade-off vis-à-vis one agent conflicts with the truth-telling requirements related to the other agent. The seller would like to allocate more or less often to one given agent than what is necessary to make the other reveal. It is therefore necessary to further distort the probability of allocating the good compared to the unconstrained mechanism.

Appendix A

A.1

Note that $u_i(v_i, v_i') = u_i(v_i', v_i') + (v_i - v_i')[E_{v_j}X_i(v_i', v_j) - \alpha_a E_{v_j} X_j(v_i', v_j)]$. Then the incentive compatibility constraint is equivalent to:

$$u_i(v_i, v_i) \geq u_i(v_i', v_i') + (v_i - v_i')[E_{v_j}X_i(v_i', v_j) - \alpha_a E_{v_j}X_j(v_i', v_j)]. \quad (6)$$

Using this inequality twice, the incentive compatibility constraint is equivalent to

$$(v_i - v_i')[E_{v_j}X_i(v_i', v_j) - \alpha_a E_{v_j}X_j(v_i', v_j)] \leq u_i(v_i, v_i) - u_i(v_i', v_i') \leq (v_i - v_i')[E_{v_j}X_i(v_i, v_j) - \alpha_a E_{v_j}X_j(v_i, v_j)]. \quad (7)$$

Then the agent reveals truthfully if:

$$E_{v_j}\left[X_i(v_i', v_j) - \alpha_a X_j(v_i', v_j)\right] \leq E_{v_j}\left[X_i(v_i, v_j) - \alpha_a X_j(v_i, v_j)\right] \quad \forall\, v_i' \leq v_i \quad (\text{IC}_2)$$

(7) must hold for all v_i' and all $v_i = v_i' + \delta$ with $\delta > 0$. Since $E_{v_j}X_i(v_i, v_j) - \alpha_a E_{v_j}X_j(v_i, v_j)$ is increasing in v_i, we can take the Riemann integral. Then, the agent reveals truthfully if we also have:

$$u_i(v_i) - u_i(v_i') = \int_{v_i'}^{v_i} E_{v_j}[X_i(s, v_j) - \alpha_a X_j(s, v_j)]ds \quad \forall\, v_i' \leq v_i \quad (\text{IC}_1).$$

To complete the proof, we need to verify that (IC_1) and (IC_2) imply (6). Suppose $v_i' \leq v_i$, then given (IC_1) and (IC_2), we have:

$$u_i(v_i, v_i) = u_i(v_i', v_i') + \int_{v_i'}^{v_i} E_{v_j}[X_i(s, v_j) - \alpha_a X_j(s, v_j)]ds$$

$$\geq u_i(v_i', v_i') + \int_{v_i'}^{v_i} E_{v_j}[X_i(v_i', v_j) - \alpha_a X_j(v_i', v_j)]ds$$

$$= u_i(v_i', v_i') + (v_i - v_i')[E_{v_j}X_i(v_i', v_j) - \alpha_a E_{v_j}X_j(v_i', v_j)].$$

The seller maximizes her expected revenue (the sum of transfers) under constraints (IC_1) and (IC_2) (to induce truthtelling) and the remaining constraints (IR), (F_0) and (F_1).[18] □

A.2

The equilibrium utility $u_i(v_i)$ must be weakly increasing (by (IC_1)) and weakly convex (by (IC_2)) in v_i. Furthermore, the slope of

18 Note that the proof is similar to Myerson (1981) except that we do not provide a sufficient condition for (IR) to hold at this stage.

the reservation utility $w_i(v_i)$ is constant and equal to $-\alpha_a$. Therefore, for any mechanism A with vector of interim probabilities $X = (X_1(v), X_2(v))$ satisfying (IC$_1$)–(IC$_2$)–(F$_0$)–(F$_1$) there is at most one interval $\hat{V}_i(A) = [\hat{v}_i^a(A), \hat{v}_i^b(A)]$ such that for all $v_i \in \hat{V}_i(A)$[19]:

$$\frac{d}{dv_i} u_i(v) = \frac{d}{dv_i} w_i(v). \tag{8}$$

Furthermore, the convexity of $u_i(v_i)$ combined with the linearity of $w_i(v_i)$ implies that:

$$\frac{d}{dv_i}(u_i(v_i) - w_i(v_i)) < 0 \quad \forall\, v_i < \hat{v}_i^a(A) \quad \text{and}$$
$$\frac{d}{dv_i}(u_i(v_i) - w_i(v_i)) > 0 \quad \forall\, v_i > \hat{v}_i^b(A). \tag{9}$$

Since informational rents are costly, (IR) will bind on $\hat{V}_i(A)$, that is:

$u_i(v_i) = w_i(v_i) \quad \forall\, v_i \in \hat{V}_i(A)$.

This completes the proof. □

A.3

Fix $\hat{V}_1$ and $\hat{V}_2$. $\pi_i^{IJ}(v_i, v_j)$ is strictly increasing in v_i and v_j and $\pi_j^{IJ}(v_j, v_i) - \pi_i^{IJ}(v_i, v_j)$ is strictly decreasing in v_i and strictly increasing in v_j for all IJ. Let $a_i^{IJ}(v_j)$ be such that $\pi_i^{IJ}(a_i^{IJ}(v_j), v_j) = 0$. By differentiating this equation and from the variations of $\pi_i^{IJ}(v_i, v_j)$, we have that $a_i^{IJ}(v_j)$ is strictly decreasing in v_j for all IJ. Let $r_i^{IJ}(v_j) = \min\left\{v_i \in I \mid \pi_i^{IJ}(v_i, v_j) \geqslant 0\right\}$, it is easy to see that $r_i^{IJ}(v_j) = a_i^{IJ}(v_j)$ when $a_i^{IJ}(v_j) \in I$. It is equal to the lower boundary of I when $a_i^{IJ}(v_j)$ falls below that boundary and it is equal to the higher boundary of I when $a_i^{IJ}(v_j)$ falls above that boundary.

Let $b^{IJ}(v_j)$ be the point such that $\pi_i^{IJ}(b^{IJ}(v_j), v_j) = \pi_j^{IJ}(v_j, b^{IJ}(v_j))$. By differentiating this expression and given the variations of $\pi_i^{IJ}(v_i, v_j)$, we have that $b^{IJ}(v_j)$ is strictly increasing in v_j for all IJ. Let $h^{IJ}(v_j) = \min\left\{v_i \in I \mid \pi_i^{IJ}(v_i, v_j) \geqslant \pi_j^{IJ}(v_j, v_i)\right\}$, it coincides with $b^{IJ}(v_j)$ if $b^{IJ}(v_j)$ is in I, it is equal to the lower boundary of I when $b^{IJ}(v_j)$ falls below that boundary and it is equal to the higher boundary of I when $b^{IJ}(v_j)$ falls above that boundary. Given the symmetry of the virtual surplus functions, we also have:

(i) $r_i^{\underline{V}_i\underline{V}_j}(v_j)$ and $r_j^{\underline{V}_i\underline{V}_j}(v_i)$; $r_i^{\hat{V}_i\hat{V}_j}(v_j)$ and $r_j^{\hat{V}_i\hat{V}_j}(v_i)$; $r_i^{\overline{V}_i\overline{V}_j}(v_j)$ and $r_j^{\overline{V}_i\overline{V}_j}(v_i)$; $r_i^{\underline{V}_i\hat{V}_j}(v_j)$ and $r_j^{\hat{V}_i,\underline{V}_j}(v_i)$; $r_i^{\underline{V}_i\overline{V}_j}(v_j)$ and $r_j^{\overline{V}_i\underline{V}_j}(v_i)$; $r_i^{\hat{V}_i\overline{V}_j}(v_j)$ and $r_j^{\overline{V}_i\hat{V}_j}(v_i)$ are symmetric.

(ii) $h^{\underline{V}_i\hat{V}_j}(v_j)$ and $h^{\hat{V}_i\underline{V}_j}(v_i)$; $h^{\underline{V}_i\overline{V}_j}(v_j)$ and $h^{\overline{V}_i\underline{V}_j}(v_i)$; $h^{\hat{V}_i\overline{V}_j}(v_j)$ and $h^{\overline{V}_i\hat{V}_j}(v_i)$ are symmetric. Moreover $h^{IJ}(v_j) = v_j$ for $IJ = \underline{V}_i\underline{V}_j$, $\hat{V}_i\hat{V}_j$, $\overline{V}_i\overline{V}_j$.

Consider the mechanism $A^{UNC}(\hat{V}_i, \hat{V}_j)$ such that the seller allocates the good to i if $v_i \geqslant \max\{r_i^{IJ}(v_j), h^{IJ}(v_j)\}$ when $v_i \in I$ and $v_j \in J$ and keeps it otherwise. This mechanism maximizes $\mathcal{P}^{UNC}(\hat{V}_i, \hat{V}_j)$. Let $\hat{V}_i'$ be the set of types such that $H(v_i) = -\alpha_a$ and let $\Psi_i(\cdot)$ be the mapping from elements $\hat{V}_i$ into elements $\hat{V}_i'$. The mechanism $A^{UNC}(\hat{V}_i, \hat{V}_j)$ is candidate for optimality if (IC$_2$) is satisfied and $\Psi_i(\hat{V}_i) = \hat{V}_i$ for all i.

- We first show that the mechanism does not satisfy (IC$_2$) at $\hat{v}_i^a$ and $\hat{v}_i^b$: by inspection of $\pi_i^{IJ}(v_i, v_j)$, $\pi_j^{IJ}(v_i, v_j)$ and $\pi_j^{IJ}(v_j, v_i) - \pi_i^{IJ}(v_i, v_j)$, we have

$$h^{\underline{V}_i\cdot}(v_j) < h^{\hat{V}_i\cdot}(v_j) < h^{\overline{V}_i\cdot}(v_j), \qquad r_i^{\underline{V}_i\cdot}(v_j) < r_i^{\hat{V}_i\cdot}(v_j) < r_i^{\overline{V}_i\cdot},$$
$$r_j^{\underline{V}_i\cdot}(v_i) < r_j^{\hat{V}_i\cdot}(v_i) < r_j^{\overline{V}_i\cdot}(v_i)$$

  implying that $H(\hat{v}_i^{a^-}) > H(\hat{v}_i^{a^+})$. The same applies at $\hat{v}_i^b$, therefore (IC$_2$) is not satisfied at $\hat{v}_i^a$ and $\hat{v}_i^b$.
- We now show that (IC$_2$) is satisfied everywhere else. Let $a_j^{IJ^{-1}}(v_j)$ be the inverse function of $a_j^{IJ}(v_i)$ (which exists and is strictly decreasing because $a_j^{IJ}(v_i)$ is strictly decreasing). The two curves $a_i^{IJ}(v_j)$ and $a_j^{IJ^{-1}}(v_j)$ cross at $\breve{v}^{IJ} = (\breve{v}_i^{IJ}, \breve{v}_j^{IJ})$. This point is unique[20] because $\frac{d}{dv_j} a_j^{IJ^{-1}}(v_j) \geqslant -1$ and $b^{IJ}(\breve{v}_j^{IJ}) = a_i^{IJ}(\breve{v}_j^{IJ})$.

  Suppose that $\breve{v}^{IJ} \in IJ$. When $v_i < \breve{v}_i^{IJ}$, i never obtains the good and j gets it if $v_j > r_j^{IJ}(v_i)$. Given the reserve price decreases in v_i, then $X_j^{A^{UNC}}(v_i', v_j) > X_j^{A^{UNC}}(v_i, v_j)$ when $v_i' > v_i$. When $v_i \geqslant \breve{v}_i^{IJ}$, i is allocated the good when $v_j \in (r_j^{IJ^{-1}}(v_i), h^{IJ^{-1}}(v_i))$ and j gets it if $v_j > h^{IJ^{-1}}(v_i)$. Given the properties of r_j^{IJ} and h^{IJ}, we have necessarily $X_i^{A^{UNC}}(v_i', v_j) - X_i^{A^{UNC}}(v_i, v_j) > -\left(X_j^{A^{UNC}}(v_i, v_j) - X_j^{A^{UNC}}(v_i', v_j)\right)$. Then $\int_J X_i^{A^{UNC}}(v) - \alpha_a \int_J X_j^{A^{UNC}}(v)$ increases in v_i. Suppose now that $\breve{v}^{IJ} \notin IJ$, either or both agents face now a decreasing reserve price (rather than strictly decreasing) but the allocation is otherwise qualitatively similar and the same argument holds for any possible sub cases of this limit case. The same is true for all I and J and overall $H(v_i)$ increases in v_i at any point but at $\hat{v}_i^a$ and $\hat{v}_i^b$.
- Last, as a consequence of the two previous points, $\Psi_i(\hat{V}_i) \neq \hat{V}_i$. □

A.4

The proof has two parts.

Part 1—we first construct the mechanism that yields highest revenue for given $\hat{V}_i$ and $\hat{V}_j$. Note that $\hat{C}$ contains all the points that generate violations of (IC$_2$) due to inconsistent allocations to agents 1 and 2 in the unconstrained mechanism. Also, at any point below (respectively above) $\hat{C}$ and below (respectively above) the 45° line, the seller always prefer to allocate the good to i (respectively to j) or to nobody. For all v_i, we define the following sets:

$$\hat{C}(v_i) = \left[\min\{\hat{c}_l(v_i), v_i\}, \max\{\hat{c}_h(v_i), v_i\}\right]$$

$$\underline{\hat{C}}(v_i) = \left[\min\{\hat{c}_l(v_i), v_i\}, v_i\right] \qquad \hat{\overline{C}}(v_i) = \left[v_i, \max\{\hat{c}_h(v_i), v_i\}\right]$$

$$\overline{D}(v_i) = \left[v_i, \overline{v}\right], \qquad \overline{C}(v_i) = \overline{D}(v_i) \setminus \hat{C}(v_i)$$

$$\underline{D}(v_i) = \left[\underline{v}, v_i\right], \qquad \underline{C}(v_i) = \underline{D}(v_i) \setminus \hat{C}(v_i).$$

$\hat{C}(v_i)$ is the set of all v_j such that v belongs to $\hat{C}$ where $\underline{\hat{C}}(v_i)$ are those below the 45° line while $\hat{\overline{C}}(v_i)$ are those above the 45° line, $\underline{C}(v_i)$ is the set of all v_j such that v lies below $\hat{C}$ and the 45° line; last, $\overline{C}(v_i)$ is the set of all v_j such that v lies above $\hat{C}$ and the 45° line. Define the operator $E_B p(v) = \int_{v_j \in B} \frac{p(v)}{\Delta} dv_j$ where $p(v)$ is a function

[19] Naturally, it may be that $du_i(v_i)/dv_i > dw_i(v_i)/dv_i$ for all v_i or $du_i(v_i)/dv_i < dw_i(v_i)/dv_i$ for all v_i.

[20] Note that $\frac{d}{dv_j} a_j^{IJ^{-1}}(v_j) = \frac{\alpha_a h}{h' - \alpha_b}$ where $h = 1 + 1_{J=\overline{V}_j} + 1_{J=\underline{V}_j}$ and $h' = 1 + 1_{I=\overline{V}_i} + 1_{I=\underline{V}_i}$.

of v. Define:

$$\hat{H}(v_i) = E_{\hat{C}(v_i)}X_i(v_i, v_j) - \alpha_a E_{\hat{C}(v_i)}X_j(v_i, v_j),$$

$$\hat{\overline{h}}(v_i) = E_{\hat{\overline{C}}(v_i)}X_i(v_i, v_j) - \alpha_a E_{\hat{\overline{C}}(v_i)}X_j(v_i, v_j)$$

$$\underline{\hat{h}}(v_i) \equiv E_{\underline{\hat{C}}(v_i)}X_i(v_i, v_j) - \alpha_a E_{\underline{\hat{C}}(v_i)}X_j(v_i, v_j)$$

$$\overline{H}(v_i) = -\alpha_a E_{\overline{C}(v_i)}X_j(v_i, v_j), \qquad \underline{H}(v_i) = E_{\underline{C}(v_i)}X_i(v_i, v_j).$$

By construction $H(v_i) = \hat{H}(v_i) + \overline{H}(v_i) + \underline{H}(v_i)$ and $\hat{H}(v_i) = \hat{\overline{h}}(v_i) + \underline{\hat{h}}(v_i)$.

Step 1: We construct the optimal mechanism in the complement of $\hat{C}$. We start with the values lying below the 45° line. Fix an allocation elsewhere such that $\hat{H}(v_i)+\overline{H}(v_i) < -\alpha_a$ for all $v_i \leqslant v_i^a$, and $\hat{H}(v_i)+\overline{H}(v_i) \leqslant -\alpha_a$ for all $v_i \in (\hat{v}_i^a, \hat{v}_i^b)$ (otherwise we cannot design a feasible allocation).

- Consider $v_i \in (\hat{v}_i^a, \hat{v}_i^b)$. The value of $\underline{H}(v_i)$ is uniquely defined. Given the virtual surplus is increasing in v_j, there exists a value $m(v_i)$ such that it is optimal to set $X_i(v_i, v_j) = 0$ for all $v_j < m(v_i)$ and $X_i(v_i, v_j) = 1$ for all $v_j \in \underline{C}(v_i)$ and $v_j > m(v_i)$. Formally, $\underline{H}(v_i) = (\min(v_i, \hat{c}_l(v_i)) - m(v_i))/\Delta = -\alpha_a - \hat{H}(v_i) - \overline{H}(v_i)$.
- Consider $v_i \in [\underline{v}, \hat{v}_i^a)$. In that region, $H(v_i)$ must be increasing in v_i and such that $H(v_i) < -\alpha_a$. In the unconstrained allocation, the seller gives the good to i when $v_j > r_i^{\underline{V}_i\underline{V}_j^{-1}}(v_i)$ and to nobody otherwise. Depending on the allocation in $\hat{C}(v_i)$ and $\overline{C}(v_i)$, the seller needs to distort her preferred choices if (i) $m(v_i) > r_i^{\underline{V}_i\underline{V}_j^{-1}}(v_i)$, because this would imply $\underline{H}(v_i) \geqslant -\alpha_a$ and (ii) if $r_i^{\underline{V}_i\underline{V}_j^{-1}}(v_i)$ and $m(v_i)$ cross several times, because this would imply $\underline{H}(v_i)$ is non monotonic. Therefore the seller selects an allocation such that the overall $H(v_i)$ increases without exceeding $-\alpha_a - \epsilon$. Given everything but the reserve price faced by i has been fixed, it is enough to look for a solution as close as possible to the curve $H(v_i)$. Therefore, we can use the procedure in Guesnerie and Laffont (1984) up to our extra constraint $H(v_i) \leqslant -\alpha_a - \epsilon$. The optimal allocation yields a piecewise weakly increasing function $H(v_i)$ that coincides with $H^{\underline{V}_i\underline{V}_j}(v_i)$ (in which case the reserve price is $r_i^{\underline{V}_i\underline{V}_j}(v_j)$) except on N disjoint intervals $[v_o^n, v_1^n]$ increasing in n where $H(v_i) = -\alpha_a - \epsilon_n$ with ϵ_n decreasing in n (in those intervals, the reserve price is slightly above $m(v_i)$).
- Consider $v_i \in (\hat{v}_i^b, \overline{v}]$. In that region $H(v_i)$ must be increasing and such that $H(v_i) > -\alpha_a$. Allocating the good to i when $v_j > r_i^{\overline{V}_iJ^{-1}}(v_i)$ ($J = \underline{V}_j, \hat{V}_j, \overline{V}_j$) does not conflict with (IC$_2$). The only concern is to give the item sufficiently often to make sure $H(v_i) > -\alpha_a$. Therefore, the seller may decide to give more than optimal. For each J, there exists a function $\tilde{r}^{\overline{V}_iJ}(v_i) < r_i^{\overline{V}_iJ^{-1}}(v_i)$ such that $X_i(v_i, v_j) = 0$ if $v_j < \tilde{r}^{\overline{V}_iJ}(v_i)$ and $X_i(v_i, v_j) = 1$ otherwise.
- The argument is symmetric and the optimal allocation to agent j in the region above the 45° line that does not contain $\hat{C}$ has the same properties.

Step 2: We now characterize the properties of the optimal allocation in $\hat{C}$. Fix an allocation in the complement of $\hat{C}$ as well as in $\hat{\overline{C}}(v_i)$. Let $k(v_i) = \underline{H}(v_i)+\overline{H}(v_i)+\hat{\overline{h}}(v_i)$. We restrict to allocations that are feasible: if we target a given value of $\underline{\hat{h}}(v_i)$, there exists an allocation that allows to reach that value.

- In $\hat{C}$, it is optimal to sell to i for some values and to j for others. Note first that, other things being equal, it is optimal to set $X_i(v) = 0$ and $X_j(v) \geqslant 0$ when $Y(v) = j$. Moreover, given the virtual surpluses are increasing in both v_i and v_j, the benefit of allocating to either agent increases in v_j. Therefore, other things being equal, it is optimal to set $X_i(v_i, v_j) \geqslant X_i(v_i, v_j')$ and $X_j(v_i, v_j) \geqslant X_j(v_i, v_j')$ when $v_j \geqslant v_j'$. Overall, other things being equal, priority should be given to the values associated with higher surplus, which should receive the good with probability 1.
- Suppose we need to distribute a given value of $\underline{\hat{h}}(v_i)$. Given that any allocation to i is weighted by 1, whereas any allocation to j is weighted down by $-\alpha_a < 1$, it is more difficult to reach the targeted $\underline{\hat{h}}(v_i)$ by allocating to j. In particular, reaching $\underline{\hat{h}}(v_i)$ may not be possible by sticking to the optimal unconstrained allocation and the seller may have to allocate to i instead of j with some probability for some values. Note that, assuming it is optimal in the unconstrained allocation to allocate the good to j at a given point, it is best to allocate to j with probability x and i with probability $1-x$ rather than allocating to i with probability 1. Therefore, the seller may randomize between the bidders.
- When $v_i \in (\hat{v}_i^a, \hat{v}_i^b)$, we must have $\underline{\hat{h}}(v_i) = -\alpha_a - k(v_i)$ (provided this quantity is positive), and the valuations associated with the highest surplus obtain the good up to the point $\underline{\hat{h}}(v_i) = -\alpha_a - k(v_i)$. If this point is not reached, the seller must allocate the item to i with at least some probability when it is best to allocate to j. When $v_i < \hat{v}_i^a$, the valuations associated with the highest surplus obtain the good provided $\underline{\hat{h}}(v_i)$ increases in v_i and lies strictly below $-\alpha_a - k(v_i)$. When $v_i > \hat{v}_i^b$, the valuations associated with the highest surplus obtain the good provided $\underline{\hat{h}}(v_i)$ increases in v_i and lies strictly above $-\alpha_a - k(v_i)$. The seller may also need to allocate to i with positive probability when it would be optimal to allocate to j to guarantee $\underline{\hat{h}}(v_i)$ has the required property.

Step 3: The characterization obtained in step 1 holds for every allocation in $\hat{C}$. The characterization obtained in step 2 holds for every allocation in the complement of $\hat{C}$. Therefore, both sets of properties form an equilibrium. For arbitrary $\hat{V}_i$ and $\hat{V}_j$, the best mechanism satisfies those properties.

Part 2—we now describe the optimal mechanism.

Step 1. The optimal mechanism is the one that yields highest revenue among all the best mechanisms described in Part 1—Step 3. Given all share the same properties, the optimal mechanism shares them as well.

Step 2: We now show that the equilibrium sets of binding types are interior and therefore the complement of $\hat{C}$ is generically non empty. Formally, we need to prove that, at equilibrium, we do not have $\hat{v}_a^i = \underline{v}$ and $\hat{v}_b^i = \overline{v}$ for all $i = 1, 2$. Suppose the contrary holds. The unconstrained allocation is $X_i(v) = 1$ if $v_i > v_j$ and $v_i > r_i^{\hat{V}\hat{V}}(v_j)$. In the constrained allocation however, we must have $H(v_i) = -\alpha_a$ for all v_i.

- When $\alpha_a \to 0$, the optimal mechanism entails $\hat{v}_a^i = \underline{v}$ for all i. This is the case because $H(v_i) \to E_{v_j}[X_i(v)] \geqslant 0$, and $-\alpha_a \to 0$. At equilibrium, $\hat{V}_i = [\underline{v}, \hat{v}_i^b]$ for all i. When α_a decreases however, it becomes costly to have an allocation such that $H(v_i) = -\alpha_a$ for low values of v_i as it requires to give the good to j and or i when this yields negative surplus. If $\hat{v}_a^i > \underline{v}$, inefficient trades when $v_i < \hat{v}_i^a$ can be avoided as we only need to satisfy $H(v_i) < -\alpha_a$. Overall, increasing $\hat{v}_a^i$ by ϵ increases the overall revenue. Therefore, at equilibrium, $\hat{v}_a^i > \underline{v}$ when $\alpha_a < 0$.
- Similarly, when $\alpha_a \to -1$, the optimal mechanism entails $\hat{v}_b^i = \overline{v}$ for all i. This occurs because $H(v_i) \to E_{v_j}[X_i(v) + X_j(v)] \leqslant 1$, and $-\alpha_a \to 1$. At equilibrium, $\hat{V}_i = [\hat{v}_i^a, \overline{v}]$ for all i. When α_a increases, it becomes costly to have an allocation such that $H(v_i) = -\alpha_a$ for high values of v_i as it requires to not allocate the good to i when this yields positive surplus. If $\hat{v}_b^i < \overline{v}$, efficient trades when $v_i > \hat{v}_i^b$ can be undertaken because we must have now $H(v_i) > -\alpha_a$. Overall, decreasing $\hat{v}_b^i$ by ϵ increases the overall revenue. Therefore, at equilibrium, $\hat{v}_b^i < \overline{v}$ when $\alpha_a > -1$. □

Appendix B

Consider the following principal–agent problem. A seller can allocate a good to an agent, keep the good or destroy the good. The valuation of the agent is $v \in [\underline{v}, \overline{v}]$ with $\underline{v} < \overline{v}$ and it is drawn from distribution $F(\cdot)$. If the seller keeps the good, she uses it and exerts a negative externality $-\alpha_a v - \gamma < 0$ on the agent. Destroying the good does not generate any value or externality. We assume Assumptions 1 and 2 hold in this setting. Moreover, suppose that $\alpha_a \in (-1, 0)$ (to make sure we are in the case where the binding type may be interior) and $\gamma \gg 0$ (to make sure that $-\alpha_a v - \gamma < 0$ for all v).

Denote by $X_1(v)$ the probability of allocating the good to the agent, $X_0(v)$ the probability of keeping the good and $t_1(v)$ the payment from the agent to the seller. The utility of the agent if he reports v' is

$$u(v, v') = vX_1(v') - \alpha_a v X_0(v') - t_1(v').$$

Incentive compatibility requires for all $v \geqslant v'$:

$$u_1(v) - u_1(v') = \int_{v'}^{v} [X_1(s) - \alpha_a X_0(s)]ds \quad (\text{IC}_1)$$

$$X_1(v) - \alpha_a X_0(v) \geqslant X_1(v') - \alpha_a X_0(v') \quad (\text{IC}_2).$$

Given Assumption 2, the worst outside option is obtained when the seller keeps the good and therefore, individual rationality requires $u(v) \geqslant w(v) = -\alpha_a v - \gamma$. Given $\frac{du}{dv} = X_1(v) - \alpha_a X_0(v)$ and $\frac{dw}{dv} = -\alpha_a$, there exists at most a set of types $\hat{V} = [\hat{v}^a, \hat{v}^b]$ such that $\frac{du}{dv} = -\alpha_a$ for all $v \in \hat{V}$. For any set of binding types $\hat{V}$, the expected revenue of the seller is

$$\int_{\underline{v}}^{\hat{v}^a} \left[X_1(v)\left[v + \frac{F(v)}{f(v)}\right] - X_0(v)\left[\alpha_a(v + \frac{F(v)}{f(v)}) + \gamma\right]\right] dF(v)$$
$$+ \int_{\hat{v}^a}^{\hat{v}^b} \Big[X_1(v)v_1 - X_0(v)[\alpha_a v + \gamma]\Big] dF(v)$$
$$+ \int_{\hat{v}^b}^{\overline{v}} \left[X_1(v)\left[v - \frac{1-F(v)}{f(v)}\right]\right.$$
$$\left. - X_0(v)\left[\alpha_a(v - \frac{1-F(v)}{f(v)}) + \gamma\right]\right] dF(v)$$
$$- F(\hat{v}^a)w(\hat{v}^a) - (1 - F(\hat{v}^b))w(\hat{v}^b) - \int_{\hat{v}^a}^{\hat{v}^b} w(v)dF(v).$$

The problem of the seller is to maximize the expected revenue under the constraints (IC_2) and $X_1(v) - \alpha_a X_0(v) = -\alpha_a$ for all $v \in \hat{V}$.

Note that all virtual surpluses are increasing in v. Let $\underline{r} = \min\{v | v + \frac{F(v)}{f(v)} \geqslant 0\}$ (an interior solution satisfies $v + \frac{F(v)}{f(v)} = 0$) and $\overline{r} = \min\{v | v - \frac{1-F(v)}{f(v)} \geqslant 0\}$ (an interior solution satisfies $v - \frac{1-F(v)}{f(v)} = 0$), we have $\underline{r} \leqslant \overline{r}$ (and $\underline{r} = \underline{v}$ if $\underline{v} \geqslant 0$). Note also that $v + \frac{F(v)}{f(v)} + \alpha_a(v + \frac{F(v)}{f(v)}) + \gamma > 0$, $v + \alpha_a v + \gamma > 0$ and $v - \frac{1-F(v)}{f(v)} + \alpha_a(v - \frac{1-F(v)}{f(v)}) + \gamma > 0$. Therefore, and other things being equal, the seller prefers to give the good to the agent rather than keeping it.

Consider a mechanism such that $X_1(v) = X_0(v) = 0$ if $v < \underline{r}$ and $X_1(v) = 1$ if $v > \overline{r}$. This implies that $\hat{V} \subset (\underline{r}, \overline{r})$. Now, for all $v \in (\underline{r}, \hat{v}^a)$, the seller's revenue is maximized if she gives the good to the agent ($X_1(v) = 1$) and when $v \in (\hat{v}^b, \overline{r})$, the seller's revenue is maximized if she destroys the good ($X_1(v) = X_0(v) = 0$). This solution is not incentive compatible.

Given keeping the good is never beneficial, let us restrict the attention to solutions such that $X_0(v) = 0$. In that class, the optimal solution is $X_1(\hat{v}) = -\alpha_a - \epsilon$ for all $v_1 \in (\underline{r}, \hat{v}^a)$, $X_1(\hat{v}) = -\alpha_a$ for all $v \in (\hat{v}^a, \hat{v}^b)$ and $X_1(\hat{v}) = -\alpha_a + \delta$ for all $v \in (\hat{v}^b, \overline{r})$ where $\epsilon \to 0$ and $\delta \to 0$. The system of transfers associated to this mechanism is:

$$t(v) = \begin{cases} \gamma - \epsilon\hat{v}^a + \underline{r}(\alpha_a + \epsilon) & v \in (\underline{v}, \underline{r}) \\ \gamma - \epsilon\hat{v}^a & v \in (\underline{r}, \hat{v}^a) \\ \gamma & v \in (\hat{v}^a, \hat{v}^b) \\ \gamma + \delta\hat{v}^b & v \in (\hat{v}^b, \overline{r}) \\ \gamma + \delta\hat{v}^b + \overline{r}(1 + \alpha_a - \delta) & v \in (\overline{r}, \overline{v}) \end{cases}$$

yielding the expected revenue[21]:

$$\gamma + F(\underline{r})[\underline{r}(\alpha_a + \epsilon) - \epsilon\hat{v}^a] - \epsilon\hat{v}^a[F(\hat{v}^a) - F(\underline{r})]$$
$$+ \delta\hat{v}^b[F(\overline{r}) - F(\hat{v}^b)] + (1 - F(\overline{r}))[\delta\hat{v}^b + \overline{r}(1 + \alpha_a - \delta)].$$

The derivatives with respect to $\hat{v}^a$ and $\hat{v}^b$ respectively are

$$-\epsilon[f(\hat{v}^a)\hat{v}^a + F(\hat{v}^a)] \propto -\epsilon\left[\hat{v}^a + \frac{F(\hat{v}^a)}{f(\hat{v}^a)}\right] < 0$$

$$-\delta[\hat{v}^b f(\hat{v}^b) - 1 + F(\hat{v}^b)] \propto -\delta\left[\hat{v}^b - \frac{1 - F(\hat{v}^b)}{f(\hat{v}^b)}\right] > 0$$

as long as $\hat{v}^a > \underline{r}$ and $\hat{v}^b < \overline{r}$. Therefore it is optimal to set $\hat{v}^a = \underline{r}$ and $\hat{v}^b = \overline{r}$. Last, it is easy to see that distorting the allocation below $\underline{r}$ or above $\overline{r}$ would decrease the seller's revenue. Moreover, increasing $X_0(v)$ on $(\underline{r}, \overline{r})$ would require decreasing the probability of a better option (allocating the good to the agent).

Overall, in the optimal mechanism, the seller never allocates the good when $v < \underline{r}$, she always allocates it when $v > \overline{r}$ and she allocates it with probability $-\alpha_a$ when $v \in (\underline{r}, \overline{r})$. She destroys the good each time it is not allocated, except when the agent does not show up. In that case, she keeps the item and exerts the negative externality. Equilibrium payments are:

$$t(v) = \begin{cases} \gamma + \underline{r}\alpha_a & v \in (\underline{v}, \underline{r}) \\ \gamma & v \in (\underline{r}, \overline{r}) \\ \gamma + \overline{r}(1 + \alpha_a) & v \in (\overline{r}, \overline{v}). \end{cases}$$

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[21] Note that the expected revenue decreases in both δ and ϵ, so these numbers must be as close as possible to 0.

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